STK2130 Trial Exam 2022

Anton Yurchenko-Tytarenko

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Problem 1

Consider a discrete-time Markov chain with state space $\mathcal{S} = \{0, 1, 2, 3, 4\}$ and the one-step transition probability matrix

$$\mathbf{P} = \begin{pmatrix} q & p & q & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 \\ 0.3 & 0.4 & 0.3 & 0 & 0 \\ 0 & 0.4 & 0 & 0.4 & 0.2 \\ p & q & 0.1 & 0.1 & 0.1 \end{pmatrix}.$$

a) Determine p and q and draw the state diagram of the Markov chain.

b) Compute the two-step transition probability matrix and find

$$\mathbb{P}\left(X_3 = 2,\ X_2 = 1 \mid X_0 = 1\right).$$

c) Find

$$\mathbb{P}(X_k \in \{0, 1, 2\} \text{ for some } 1 \leq k \leq 4 \mid X_0 = 3),$$

i.e. the probability that $\{X\}$ will visit $\{0, 1, 2\}$ at least once within 4 steps given that is starts at state 3.

d) Determine the classes of the Markov Chain. Give the definition of recurrent and transient states and, for each class, determine whether it is transient or recurrent. Is this Markov chain ergodic?

e) Conditioned upon the chain has entered one of the states 0, 1 or 2, find the stationary distribution over these three states.

f) Let $s_{4,3}$ denote the expected number of visits to state 3 given that $X_0 = 4$. Find $s_{4,3}$.

Problem 2

a) What does it mean that the random variable X has Poisson distribution?

b) Give **both** definitions of the homogeneous Poisson process. What does the word "*homogeneous*" mean in this context?

The café "*Siméon Denis*" is open 24 hours a day, 7 days a week. Its manager wants to analyze the number of clients arriving every day.

c) At first, the manager assumes that the clients arrive according to a homogeneous Poisson process with the rate $\lambda = 8$ clients per hour.

   1) Find the expected number of clients per day (i.e. from 00:00 to 24:00).
   2) What is the probability that there are no clients from 8:00 to 10:00?
   3) What is the probability that the café will have 3 or more customers from 8:00 to 10:00?
   4) What is the expected number of customers from 8:00 to 10:00?
   5) Let S_{20} be the time of arrival of the 20th customer. What is the distribution of S_{20}? Write its density.
   6) Find $\mathbb{E}[S_{20}]$.
   7) The café has a special menu for children. Each customer is a child with probability $p = \frac{1}{4}$. What is the probability that there are no children customers from 8:00 to 10:00? Find the expected number of children at this time period.

d) The manager noticed that that the number of clients depends on time of the day. Therefore, he decided to make his model more complicated and use a non-homogeneous Poisson process with the rate ($t = 0$ corresponds to 00:00)

$$\lambda(t) = \begin{cases} 1, & t \in [0, 6), \\ 1 + 3(t - 6), & t \in [6, 10), \\ 13, & t \in [10, 20), \\ 13 + 3(20 - t), & t \in [20, 24). \end{cases}$$

   1) Write the definition of the non-homogeneous Poisson process (either of two).
   2) Find the expected number of clients from 0:00 to 24:00.
   3) What is the probability that exactly 5 clients arrive between 8:00 and 12:00?

Problem 3

a) What characterizes a birth and death process $\{X(t),\ t \geq 0\}$ with nonnegative parameters $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\mu_n\}_{n=0}^{\infty}$?

Assume that, for any $n \geq 0$,

$$\mu_n = \mu n,$$
$$\lambda_n = \lambda(n+1),$$

for some μ and $\lambda > 0$.

b) Find $M(t) = \mathbb{E}[X(t) \mid X(0) = i]$.

c) When do the limiting probabilities exist for such a birth and death process $\{X(t),\ t \geq 0\}$? Find the limiting probabilities.

Hint.

$$M(t) = \begin{cases} \frac{\lambda}{\lambda-\mu}(e^{(\lambda-\mu)t} - 1) + ie^{(\lambda-\mu)t}, & \text{if } \lambda \neq \mu, \\ \lambda t + i, & \text{otherwise,} \end{cases}$$

Limiting probabilities exist if:

$$\frac{\lambda}{\mu} < 1$$

and are equal to

$$\pi_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right), \quad n \geq 0.$$