STK2130 Exercise session 2

February 4, 2022

1. Let $\mathbf{P}$ be the transition probability matrix of a Markov chain. Argue that if for some positive integer r, $\mathbf{P}^r$ has all positive entries, then so does $\mathbf{P}^n$, for all integers $n \geq r$.

2. Prove that if the number of states in a Markov chain is $M < \infty$, and if state j can be reached from state i, then it can be reached in M steps or less.

3. Consider a Markov chain with state space $\mathcal{S} = \{0, 1, 2, 3, 4\}$ and transition probability matrix:

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

   By direct computation of $\sum_{n=1}^{\infty} P_{i,i}^n$, find out which states are recurrent and which are transient.

4. Specify the classes of the following Markov chains, and determine whether they are transient or recurrent:

$$\mathbf{P}_1 = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}, \quad \mathbf{P}_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\mathbf{P}_3 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad \mathbf{P}_4 = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

5. In a good weather year the number of storms is Poisson distributed with mean 1; in a bad year it is Poisson distributed with mean 3. Suppose that any year's weather conditions depends on past years only through the previous year's condition. Suppose that a good year is equally likely to be followed by ither a good or a bad year, and that a bad year is twice as likely to be followed by a bad year as by a good year. Suppose that last year — call it year 0 — was a good year.

   (a) Find the expected total number of storms in the next two years (that is, in years 1 and 2).

   (b) Find the probability there are no storms in year 3.