5.82

Check the four axioms:

i) $N(0)=0$ because $X_1, X_2, \ldots$ are positive random variables.

ii) Since the X_i's are independent, and N is independent of the X_i's, $N(t)$ has independent increments.

iii) $P(N(t+h)-N(t)=1) = P\left(\sum_{i=1}^{N} t \le X_i \le t+h,\ X_j \notin (t, t+h)\ j \ne i\right)$

Note: $P(t \le X_i \le t+h) = \int_t^{t+h} f(s)ds = f(t)\cdot h + \int_t^{t+h} f(s)ds - f(t)h$
$= f(t)\cdot h + o(h)$

$\Rightarrow P\left(\sum_{i=1}^{N} t \le X_i \le t+h,\ X_j \notin (t, t+h)\ j \ne i\right) = E\left[\sum_{i=1}^{N} (f(t)h + o(h))(f(t)+o(h))^{N-1}\right]$
$= E[N(hf(t)+o(h))] = (f(t)\cdot h + o(h))E[N] = \underline{\lambda f(t)\cdot h + o(h)}$

iv) ~~$P(N(t+h) - N(t)) = E\left[\sum_{i=1}^{N}(\right.$~~
$P(N(t+h)-N(t)=k) = E\left[\sum_{i=1}^{N} (f(t)h + o(h))^k (f(t)+o(h))^{N-k}\right]$
$= E\left[\sum_{i=1}^{N} o(h)\right] = \underline{o(h)}$

5.85

a) $E[X(t) \mid X(s)] = E[X(t) - X(s) + X(s) \mid X(s)]$

$= E[X(t-s)] + X(s) = \lambda(t-s)E[X] + X(s)$

independent of $X(s)$

b) $E[X(t) \mid N(s)] = E[X(s) + (X(t) - X(s)) \mid N(s)]$

$= E[X(s) \mid N(s)] + E[X(t) - X(s)]$

$= N(s)E[X] + \lambda(t-s)E[X]$

c) $Var(X(s) \mid N(s)) + Var(X(t) - X(s))$

$= N(s) \cdot Var(X) + Var(X(t) - X(s))$

$= N(s) Var(X) + Var(X(t-s))$

$= N(s)(E[X^2] - (E[X])^2) + \lambda(t-s)E[X^2].$

d) $E[X(s) \mid N(t)] = E[X(t) - (X(t) - X(s)) \mid N(t)]$

$= E[X(t) \mid N(t)] - E[X(t) - X(s) \mid N(t)] = N(t)E[X] - \lambda(t-s)E[X]$

5.86 a) $P(N(t)=n) = 0.3\,P(N(t)=n \mid \text{good year})$
$+ 0.7\,P(N(t)=n \mid \text{not good year})$
$= 0.3\,P(N_1(t)=n) + 0.7\,P(N_2(t)=n)$
$$= 0.3\,\frac{(3t)^n}{n!}e^{-3t} + 0.7\,\frac{(5t)^n}{n!}e^{-5t}$$

b) If $N(t)$ is a Poisson Process, it must ~~have~~ be a poisson random variable with rate λt, for some $\lambda > 0$. Then:

$$0.3\,\frac{(3t)^n}{n!}e^{-3t} + 0.7\,\frac{(5t)^n}{n!}e^{-5t} = \frac{(\lambda t)^n}{n!}e^{-\lambda t},$$

but no such λ exists. Hence $N(t)$ is not a Poisson process.

c) and d)
Since both Poisson processes and the probability of a good year are independent of time, ~~the~~ $N(t)$ has stationary increments. Since both Poisson processes have independent increments, $N(t)$ has independent increments.

e) $P(\text{good year} \mid N(1)=3)$

$$= \frac{P(\text{good year} \cap N(1)=3)}{P(N(1)=3)}$$

$$= \frac{0.3 \frac{3^3}{3!} e^{-3}}{\frac{3^3}{3!} e^{-3} \cdot 0.3 + \frac{5^3}{3!} e^{-5} \cdot 0.7} \approx 0.406$$

5.87

$Cov(X(t), X(t+s))$ $\quad X(t)$ $\underbrace{}_{\text{compound}}$ Poisson process.

Since $X(t) = \sum_{i=1}^{N(t)} Y_i$ where Y_i are independent, $X(t+s) - X(t)$ and $X(t)$ are independent:

$\sum_{i=N(t)}^{N(t+s)} Y_i$, $\sum_{i=1}^{N(t)} Y_i$ are independent sums with independent limits because $N(t)$ has independent increments

Hence,

$$Cov(X(t), X(t+s)) = Cov(X(t), X(t) + X(t+s) - X(t))$$

$$= Cov(X(t), X(t)) + Cov(X(t), X(t+s) - X(t))$$

$$= Var(X(t)) + 0$$

$$= \lambda t E[Y_1^2]$$