5.64.) People arrive at bus stop by Poisson process rate λ. Bus departs at t.

$X :=$ total amount of wait for those who get at bus at t

Want to find $Var(X)$. Let $N(t) :=$ # arrivals by time t

a) $E[X \mid N(t)]$?

Each per. waits t - arrival time, there are n people arriving before bus departs

U_i's independent, identical distr.

$$E[X \mid N(t)=n] = E\left[\sum_{i=1}^{n} (t - U_i)\right] = n\,E[t - U_i]$$

Thm. 5.2: Given $N(t)=n$, events are uniformly distr. in $[0, t]$.

Let $U_i \sim unif(0,t)$ be the arrival of the i'th person out of total n

NB: does not actually have to arrive as # i; unordered

$N(t)=n$

$$= N(t)\left(t - \frac{t}{2}\right)$$

$$= \underline{\underline{N(t)\,\frac{t}{2}}}$$

③

b) $Var(X | N(t))$?

$$Var(X | N(t)=n) = Var\left(\sum_{i=1}^{n} (t-U_i)\right)$$

$$= N(t)\ Var(t-U_i)$$

$$= N(t)\left(Var(t) + Var(-U_i)\right)$$

$$= N(t)\left(0 + (-1)^2 Var(U_i)\right)$$

$$= N(t)\frac{1}{12}(t-0)^2$$

$$= N(t)\frac{t^2}{12}$$

Thm. 5.2: Given $N(t)=n$, events (unordered) are uniformly distributed in $[0,t]$. Let $U_i \sim unif(0,t)$ be the arrival of person i out of n (does not actually have to arrive as # i since unordered)

U_i's independent, so var (sum) = sum of variances

c) $Var(X)$?

$$Var(X) = E[Var(X | N(t))] + Var(E[X | N(t)])$$

conditional variance formula

$$= E[N(t)\frac{t^2}{12}] + Var\left(\frac{t}{2} N(t)\right)$$

a) & b)

$$= \frac{t^2}{12} E[N(t)] + \frac{t^2}{4} Var(N(t))$$

$$= \frac{t^2}{12}\lambda t + \frac{t^2}{4}\lambda t$$

$$= \lambda t^3 \left(\frac{1}{12} + \frac{1}{4}\right) = \lambda t^3 \frac{1+3}{12} = \frac{\lambda t^3}{3}$$

$N(t)$ is Poisson r. v. with mean and variance λt

78.

$$\lambda(t) = \begin{cases} 4 & \text{from } 8 \text{ til } 10 \\ 8 & \text{from } 10 \text{ to } 12 \\ 8+t & \text{from } 12 \text{ to } 14 \\ 10-2t & \text{from } 14 \text{ to } 17 \end{cases}$$

We have a non-homogeneous Poisson process with density function defined above. We will find the total expected ~~no~~ number of customers during one day:

$$E[\text{Total customers}] = E[\text{Customers between 8 and 10}]$$
$$+ E[\text{customers between 10 and 12}] + \dots$$
$$= 2\cdot 4 + 2\cdot 8 + \int_0^2 8+t\,dt + \int_0^3 10-2t\,dt$$
$$= 8 + 16 + 18 + 21 = 63$$

Since the sum of Poisson distributed variables is itself Poisson distributed $P = P_1 + P_2 + \dots$ with $\lambda = \lambda_1 + \lambda_2 + \dots$, the total amount of customers is Poisson distributed with $\lambda = 63$, (since $E[\text{total customers}] = \lambda = 63$).

80\. a) a nonhomogeneous Poisson process still has independent increment, so the T_i are independent.

b) No. Say $\lambda(t)$ increases with time, for example $\lambda(t) = t$, then ~~we~~ the expected value $E[T_i]$ decreases with i, hence the T_i are not identically distributed.

c) We write an expression for the CDF for T_1:

$$P(T_1 \leq t) = 1 - P(N(t) = 0) = 1 - e^{-\int_0^t \lambda(y)\,dy}$$

The last equation is true from Lemma 5.3 in the book (page 339). To find the PDF of T_1, we differentiate this:

$$\frac{\partial P(T_1 \leq t)}{\partial t} = \frac{\partial\left(-e^{-\int_0^t \lambda(y)\,dy}\right)}{\partial t} = \frac{\partial \int_0^t \lambda(y)\,dy}{\partial t} e^{-\int_0^t \lambda(y)\,dy}$$

$$= \lambda(t)\, e^{-\int_0^t \lambda(y)\,dy}$$

This is a nonhomogeneous exponential distribution with the same intensity function as $N(t)$.

Exercise 3

a The arrival times are exponential, so $f(t) = \lambda e^{-\lambda t} = 4e^{-4t}$.

b The service time, in minutes, is $t = 1/12$, and the number of arrivals between two successive lifts follows a Poisson with parameter $\lambda t_s = 1/3$. Therefore

$$P[N = n] = e^{-1/3}\frac{(1/3)^n}{n!}.$$

c This is the probability that two customers arrive before the first lift leaves, i.e., from above,

$$P[N = 2] = e^{-1/3}\frac{(1/3)^2}{2!}.$$

d In this case, at most one customer should arrive before the lift 1 leaves, and no one between the first and the second. I.e.,

$$(P[N = 0] + P[N = 1])P[N = 0] = [e^{-1/3} + e^{-1/3}(1/3)]e^{-1/3} = \frac{4e^{-2/3}}{3}$$