Ch. 4: Markov chains

Exercises: 2, 5, 7, 10, 11, 12

2) X_n = size of generation n

Each $\underbrace{\text{individual}}_{I}$ has $\underbrace{\text{Poisson distr. \# of offspring}}_{O_I}$:

$$P(O_I = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

RECALL:

Sum of independent Poisson r.v. with mean λ_i are Poisson with rate $\sum_i \lambda_i$.

$$X_{n+1} = X_n + Y_n \qquad (\star)$$

Assume no-one dies since not mentioned

May be accounted for in λ

where $Y_n \sim \text{Poisson}(\lambda X_n)$

This means that $\{X_n, n \geq 0\}$ is a Markov chain because the distribution of X_{n+1} only depends on X_n (distribution of future state depends only on the present state.

The transition probabilities are:

$$P(X_{n+1} = j \mid X_n = i) \underset{(\star)}{=} P(X_n + Y_n = j \mid X_n = i)$$

$$= P(i + Y_n = j \mid X_n = i) = P(Y_n = j - i \mid X_n = i) = \frac{(\lambda i)^{j-i} e^{-\lambda i}}{(j-i)!}$$

From (*): $Y_n \sim \text{Poisson}(\lambda X_n)$
$= \text{Poisson}(\lambda i)$

5.) A Markov chain $\{X_n, n \geq 0\}$ with states $S := \{0, 1, 2\}$ has transition prob. matrix:

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

Assume $P(X_0 = 0) = P(X_0 = 1) = \frac{1}{4}$. What is $E[X_3]$

<u>Know</u>:

$$E[X_3] = \sum_{x \in S} x \underbrace{P(X_3 = x)}_{\text{Need to find!}}$$

Starting distribution:

$$\big(P(X_0 = 0), P(X_0 = 1), P(X_0 = 2)\big) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$$

To go from X_0 to X_3, there are 3 steps ($X_0 \xrightarrow{1} X_1 \xrightarrow{2} X_2 \xrightarrow{3} X_3$).

From Remark <u>page 204</u> (bottom):

In general: $\underbrace{P(X_n = j)}_{\text{unconditional prob.}} = \sum_{i=0}^{\infty} P(X_n = j \mid X_0 = i)\, P(X_0 = i)$

condition on the initial state

$$= \sum_{i=0}^{\infty} \underbrace{P_{ij}^{(n)}}_{:= \text{ n-step transition probability}} \underbrace{P(X_0 = i)}_{\text{initial distribution}}$$

For our case:

$$P(X_3 = j) = \sum_{i=0}^{2} \underbrace{P_{ij}^{(3)}}_{\substack{\text{Know these} \\ \text{from transition} \\ \text{prob. matrix}}} \underbrace{P(X_0 = i)}_{\substack{\text{Know this} \\ \text{from initial distribution}}}$$

In matrix form:

$$\begin{bmatrix} P(X_3 = 0) \\ P(X_3 = 1) \\ P(X_3 = 2) \end{bmatrix}^T = \begin{bmatrix} P(X_0 = 0) \\ P(X_0 = 1) \\ P(X_0 = 2) \end{bmatrix}^T \underbrace{\mathbb{P}^{(3)}}_{\substack{\text{3-step transition} \\ \text{matrix } = \mathbb{P}^3 \text{ by} \\ \text{induction (see pg. 197)}}}$$

ex. text

$$= \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{2} \end{bmatrix}^T \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}^3$$

Matlab/ Python/ hard labor

$$= \begin{bmatrix} \frac{59}{144} \\ \frac{43}{216} \\ \frac{169}{432} \end{bmatrix}$$

Hence:

$$E[X_3] = \sum_{x=0}^{2} P(X_3 = x)\, x$$

$$= 0 \cdot \frac{59}{144} + 1 \cdot \frac{43}{216} + 2 \cdot \frac{169}{432}$$

$$\approx 0{,}9815$$

7.) Example 4.4: Suppose that it rained neither yesterday nor the day before yesterday. What's the prob. of rain tomorrow?

NOTE:

Don't know anything about today.

- Therefore: Assume we are looking at this as of yesterday.

- Then: It has not rained "yesterday" or "today" (actually two days ago & yesterday, but we have shifted the time) ⇒ Are in state 3, so

$$X_0 = 3$$

- We want to know

$$P(\text{rain tomorrow})$$

- This can happen via two states: $X_2 = 0$ or $X_2 = 1$, so

$$P(\text{rain tomorrow}) = P(X_2 = 0 \mid X_0 = 3) + P(X_2 = 1 \mid X_0 = 3)$$

$$= P_{3,0}^{(2)} + P_{3,1}^{(2)}$$

Two steps from X_0 to X_2

Find $\mathbb{P}^{(2)} = \mathbb{P}^2$:

$$\mathbb{P}^2 = \begin{bmatrix} 0{,}7 & 0 & 0{,}3 & 0 \\ 0{,}5 & 0 & 0{,}5 & 0 \\ 0 & 0{,}4 & 0 & 0{,}6 \\ 0 & 0{,}2 & 0 & 0{,}8 \end{bmatrix} \begin{bmatrix} 0{,}7 & 0 & 0{,}3 & 0 \\ 0{,}5 & 0 & 0{,}5 & 0 \\ 0 & 0{,}4 & 0 & 0{,}6 \\ 0 & 0{,}2 & 0 & 0{,}8 \end{bmatrix}$$

go from i to 0, i=...

$$= \begin{bmatrix} \frac{49}{100} & \frac{3}{25} & \frac{21}{100} & \frac{9}{50} \\ \frac{7}{20} & \frac{1}{5} & \frac{3}{20} & \frac{3}{10} \\ \frac{1}{5} & \frac{3}{25} & \frac{1}{5} & \frac{12}{25} \\ \frac{1}{10} & \frac{4}{25} & \frac{1}{10} & \frac{16}{25} \end{bmatrix}$$

go from 3 to i, i=...

Hence, $P_{3,0}^{(2)} = \frac{1}{10}$

$P_{3,1}^{(2)} = \frac{4}{25}$

So, $P(\text{rain tomorrow}) = \frac{1}{10} + \frac{4}{25}$

$$= \frac{25+40}{250} = \frac{65}{250}$$

$$= \frac{13}{50} = \underline{\underline{0{,}26}}$$

10.) Example 4.3: Gary is currently cheerful. What is the prob. that he is not in a glum mood the next 3 days?

Gary's mood, $\{X_n, n \geq 0\}$, is a 3-state Markov chain $S = \{0, 1, 2\}$ with

$$\begin{array}{ccc} 0 & 1 & 2 \\ \shortparallel & \shortparallel & \shortparallel \\ C & S & G \end{array}$$

From Example 4.3

$$\mathbb{P} = \begin{bmatrix} 0{,}5 & 0{,}4 & 0{,}1 \\ 0{,}3 & 0{,}4 & 0{,}3 \\ 0{,}2 & 0{,}3 & 0{,}5 \end{bmatrix}$$

We want to find

P(Gary not glum next 3 days | cheerful now)

$$= P(X_n \neq G,\ n=1,2,3 \mid X_0 = C)$$

To find this probability, we use the method on pg. 200 (bottom) in Ross:

Let $A := \{G\}$. Then,

$$(\star)\begin{cases} P(X_k \neq G,\ k=1,2,3 \mid X_0 = C) = 1-\beta, \text{ where} \\ \beta = P(X_k \in A \text{ for some } k=1,2,3 \mid X_0 = C) \end{cases}$$

METHOD PG 200:

Define new Markov chain $\{W_n\}$:

$$N := \begin{cases} \min\{n : X_n \in A\}, & \text{if } X_n \in A \text{ for some } n \\ \infty, & \text{if } X_n \notin A\ \forall n \end{cases}$$

$$W_n := \begin{cases} X_n & \text{if } n < N \\ A & \text{if } n \geq N \end{cases}$$

A ↳ additional state to capture X_n entering A

Then, $\{W_n\}$ is a Markov chain with states i, $i \notin A$ and A. The transition probabilities, Q_{ij}, are:

$$Q_{ij} = \underbrace{P_{ij}}_{\text{trans. prob for } \{X_n\}} \quad \text{if } i,j \notin A$$

$$Q_{i,A} = \sum_{j \in A} P_{i,j} \quad \text{if } i \notin A$$

$$Q_{A,A} = 1$$

Then:

$$P(X_k \in A \text{ for some } k = 1,2,3 \mid X_0 = C)$$

$$= P(W_3 = A \mid X_0 = C)$$

since X_k enters A by time 3 iff. W_3 is in state A (per def. of $\{W_n\}$)

$$= P(W_3 = A \mid W_0 = C)$$

def $\{W_n\}$

$$= Q^{(3)}_{C,A}$$

W goes from C to A in 3 steps

So, from ($\star$):

$$P(X_k \neq G,\ k = 1,2,3 \mid X_0 = C) = 1 - P(X_k \in A \text{ for some } k = 1,2,3 \mid X_0 = C)$$

$$= 1 - Q^{(3)}_{C,A}$$

go from $i = C, S, G$ to G

go from C to $i = C, S, G$

Know from Ex. 4.3: $$\mathbb{P} = \begin{bmatrix} 0,5 & 0,4 & 0,1 \\ 0,3 & 0,4 & 0,3 \\ 0,2 & 0,3 & 0,5 \end{bmatrix}$$

Hence, $\{Q_n\}$ has states $\{C, S, A\}$ and

$$Q = \begin{bmatrix} 0,5 & 0,4 & 0,1 \\ 0,3 & 0,4 & 0,3 \\ 0 & 0 & 1 \end{bmatrix}$$

Same as $\mathbb{P}$ since only one element in $A; G$

Go from A to A, S, C

go from $i = A, S, C$ to A

$$Q^3 = \begin{array}{c} \\ C \\ S \\ A \end{array} \begin{array}{c} \begin{array}{ccc} C & S & A \end{array} \\ \begin{bmatrix} 0,293 & 0,292 & 0,415 \\ 0,219 & 0,22 & 0,561 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

matlab/ python/ manual labor

Hence, $Q^{(3)}_{C,A} = 0,415$ and the prob. of Gary not being glum in the next 3 days is $1 - 0,415 = 0,585$

11.) Example 4.13: What are the transition probabilities of the Y_n Markov chain in terms of the X_n chain transition probabilities $(P_{i,j})$?

o—o

Example 4.13: $\{X_n, n \geq 0\}$, $S = \{0, 1, 2, 3\}$

Interested in when pattern 1, 2, 1, 2 appears.

To find this, def. new Markov Chain $\{Y_n, n \geq 0\}$:

- If pattern appeared by n'th transition; $Y_n = 4$
- If pattern has not appeared by n'th transition

$Y_n = 1$ if $X_n = 1$ and $(X_{n-2}, X_{n-1}) \neq (1,2)$

$Y_n = 2$ if $X_{n-1} = 1$, $X_n = 2$

$Y_n = 3$ if $X_{n-2} = 1$, $X_{n-1} = 2$, $X_n = 1$

$Y_n = 5$ if $X_n = 2$, $X_{n-1} \neq 1$

$Y_n = 6$ if $X_n = 0$

$Y_n = 7$ if $X_n = 3$

$\{Y_n, n \geq 0\}$ is a Markov chain with states
$S = \{1, 2, \dots, 7\}$.

Want to find, $Q_{i,j}$, transition prob. for $\{Y_n\}$,
$i, j = 1, 2, \dots, 7$:

$$Q_{i,j} = P(Y_{n+1} = j \mid Y_n = i)$$

Know $\{X_n, n \geq 0\}$ is Markov chain with 4 states $\{0, 1, 2, 3\}$ & transition Prob. $P_{i,j}$, $i, j = 0, 1, 2, 3$.

In theory, we need to find:

$$\begin{matrix} Q_{1,1}, & Q_{1,2}, & \dots, & Q_{1,7} \\ Q_{2,1}, & Q_{2,2}, & \dots, & Q_{2,7} \\ \vdots & & & \\ Q_{7,1}, & \dots & , & Q_{7,7} \end{matrix}$$

, $7 \times 7 = 49$ probabilities

Take two as examples of method:

$$Q_{1,1} = P(Y_{n+1} = 1 \mid Y_n = 1)$$

$$= P(Y_{n+1} = 1 \mid X_n = 1 \text{ and } (X_{n-2}, X_{n-1}) \neq (1,2))$$

$$= P(X_{n+1} = 1 \text{ and } (X_{n-1}, X_n) \neq (1,2) \mid X_n = 1 \text{ and } (X_{n-2}, X_{n-1}) \neq (1,2))$$

$$= P(X_{n+1} = 1 \mid X_n = 1 \text{ and } (X_{n-2}, X_{n-1}) \neq (1,2))$$

Know $X_n = 1 \neq 2$,
so $(X_{n-1}, X_n) \neq (1,2)$

$$= P(X_{n+1} = 1 \mid X_n = 1) = P_{1,1}$$

Markov
property: only
present state matters

$$Q_{3,4} = P(Y_{n+1} = 4 \mid Y_n = 3)$$

$$= P(\text{pattern appeared by } (n+1)\text{'th step} \mid X_{n-2} = 1, X_{n-1} = 2, X_n = 1)$$

$$= P(X_{n+1} = 2 \mid X_n = 1) = P_{1,2}$$

Markov prop:
only present state matters

... and so on for the
remaining 47 probabilities...

12) Markov chain $\{X_n, n \geq 0\}$, $P_{i,j}$.

Conditional prob. of $X_n = m$ given $X_0 = i$ and has not entered r by time n, where $r \neq i, m$.

Q: Is this cond. prob. equal the n-stage transition prob. of a Markov Chain whose state space does not include state r and whose transition prob. are

$$Q_{ij} = \frac{P_{ij}}{1 - P_{i,r}}, \quad i, j \neq r$$

CLAIM: Prove $P(X_n = m \mid X_0 = i, X_k \neq r, k = 1, \dots, n\} = Q_{i,m}^{n}$

or make counterexample.

Counterexample:

Consider Exercise 4.10:

Gary's mood:

$$\begin{array}{c} \\ C \\ S \\ G \end{array} \begin{array}{c} \begin{array}{ccc} C & S & G \end{array} \\ \begin{bmatrix} 0,5 & 0,4 & 0,1 \\ 0,3 & 0,4 & 0,3 \\ 0,2 & 0,3 & 0,5 \end{bmatrix} \end{array}$$

Check claim for

$P(X_4 = S \mid X_i \neq G, i = 1, 2, 3, X_0 = C)$.

Using the absorbing state method from Exercise 10, we find the four-step transition matrix for the new Markov chain $\{W_n\}$:

$$Q^{(4)} = Q^4 = \begin{bmatrix} 0,5 & 0,4 & 0,1 \\ 0,3 & 0,4 & 0,3 \\ 0 & 0 & 1 \end{bmatrix}^4 = \begin{bmatrix} 0,234 & 0,234 & 0,532 \\ 0,175 & 0,175 & 0,649 \\ 0 & 0 & 1 \end{bmatrix}$$

So; $Q^{(4)}_{C,S} = 0,234$

Now, use the formula given here:

$\frac{P_{i,j}}{1-P_{i,r}} = \frac{P_{C,S}}{1-P_{C,G}}$

Read off $\tilde{Q}^{(4)}_{C,S}$ from

$$\tilde{Q}^{(4)} = \tilde{Q}^4 = \begin{bmatrix} \frac{0,5}{1-0,1} & \frac{0,4}{1-0,1} \\ \frac{0,3}{1-0,3} & \frac{0,4}{1-0,3} \end{bmatrix}^4$$

only consider $i,j \neq G$, i.e., only $j,i = C,S$

$$= \begin{array}{c c} & \begin{array}{cc} C & S \end{array} \\ \begin{array}{c} C \\ S \end{array} & \begin{bmatrix} 0,49 & 0,51 \\ 0,49 & 0,51 \end{bmatrix} \end{array}$$

So $\tilde{Q}^{(4)}_{C,S} = 0,51$.

But note that;

$$P(\underbrace{X_4 = S}_{A} \mid \underbrace{X_i \neq G,\ i=1,2,3}_{B},\ \underbrace{X_0 = C}_{\overline{C}})$$

$$= \frac{P(X_4 = S,\ X_i \neq G\ i=1,2,3 \mid X_0 = C)}{P(X_i \neq G,\ i=1,2,3 \mid X_0 = C)}$$

Conditional probability

$P(A \mid B, \overline{C}) = \frac{P(A, B \mid \overline{C})}{P(B \mid \overline{C})}$

$$= \frac{Q^{(4)}_{C,S}}{1 - Q^{(3)}_{C,G}} = \frac{0,234}{1-0,415} = \underline{0,4}$$

def of $\{W_n\}$; glum is absorbing

read off matrix prev. pg.

$$Q^{(3)} = Q^3 = \begin{bmatrix} 0,293 & 0,292 & 0,415 \\ 0,219 & 0,22 & 0,561 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence,

which we know is correct

From the absorbing state method on pg. 200,

$$P(X_4 = S \mid X_i \neq G, i=1,2,3, X_0 = C) = \underline{0{,}4}$$

Known to be true

From the claim in the exercise,

$$P(X_4 = S \mid X_i \neq G, i=1,2,3, X_0 = C) = \underline{0{,}51}$$

? CLAIM ?

Not equal!

Hence, the claim is not true, and the example above is a counterexample.