Exercise 1 Let $\mathbf{P}$ be a transition probability matrix. Argue that if for some positive integer r, $\mathbf{P}^r$ has all positive entries, then so does $\mathbf{P}^n$ for all integers $n \geq r$.

Solution Let $\mathbf{P}^r$ have all positive entries. It is enough to prove that $\mathbf{P}^{r+1}$ has all positive entries.

Since $\mathbf{P}^{r+1} = \mathbf{P}^r \mathbf{P}^1$, we have that, for any $i,j \in \mathcal{S}$, $P^{r+1}_{i,j} = \sum_{s \in \mathcal{S}} P^r_{i,s} P_{s,j}$. Since $P^r_{i,s} > 0$ and $P_{s,j} \geq 0$, $P^{r+1}_{i,j} = 0$ if and only if $P_{s,j} = 0$ for all $s \in \mathcal{S}$, i.e $\mathbf{P}$ must have j-th column consisting of only zeros. However, for such $\mathbf{P}$, $P^n_{s,j} = 0$ for all $s \in \mathcal{S}$ and $n \geq 1$, i.e. for matrix $\mathbf{P}$ with zero j-th column, it is impossible that $\mathbf{P}^r$ with all positive entries.

Thus we conclude that $\mathbf{P}$ cannot have zero j-th column

$$\Rightarrow \quad \sum_{s \in \mathcal{S}} P^r_{i,s} P_{s,j} = P^{r+1}_{i,j} > 0 \qquad \blacksquare$$

Exercise 2 Prove that if the number of states in a Markov chain is $M < \infty$, and if state j can be reached from state i, then it can be reached in M steps or less.

Solution Let $\mathcal{N} := \{ n \geq 0 \mid P^n_{i,j} > 0 \}$. We know that, since $i \to j$, $\mathcal{N} \neq \emptyset$, so we can consider $M^* := \min \mathcal{N}$.

Assume that $M^* > M$. By definition of M^*, $P^{M^*}_{i,j} > 0 \implies$ there exist states $i_1, i_2, \dots, i_{M^*-1}$ such that

$$P_{i,i_1} P_{i_1,i_2} \cdot \dots \cdot P_{i_{M^*-1},j} > 0.$$

Denote $i_0 := i$ and observe that, since $M_* > M$, there must be $n, m \in \{0, 1, 2, \dots, M^*-1\}$, $m < n$, such that $i_n = i_m$. However in this case

$$P^{(M^*-n+m)}_{i,j} = P_{i_0,i_1} P_{i_1,i_2} \cdot \dots \cdot P_{i_{m-1},i_m} \cdot P_{\underbrace{i_n}_{= i_m},i_{n+1}} P_{i_{n+1},i_{n+2}} \cdot \dots \cdot P_{i_{M^*-1},j} > 0.$$

Thus $M^* - n + m \in \mathcal{N}$, but $M^* - n + m < M^*$, which contradicts to the fact that $M^* = \min \mathcal{N}$. Therefore $M^* \leq M$, which is required.

Exercise 3 Let $\{X_n\}$ be a Markov chain with state space $S = \{0,1,2,3,4\}$ and transition probability matrix

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

By direct computation of $\sum_{n=1}^{\infty} P^n_{i,i}$, find out which states are recurrent and which states are transient.

Solution In order to compute $\sum_{n=1}^{\infty} P^n_{i,i}$, we will compute P^n. Note that P has a block structure with many zero entries:

$$P = \left(\begin{array}{cc|cc|c} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \hline 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \hline \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{2} \end{array}\right),$$

so we denote: $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} =: A$, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} =: O_{2\times2}$

$\begin{pmatrix} 0 \\ 0 \end{pmatrix} =: O_{2\times1}$, $\left(\frac{1}{4}, \frac{1}{4}\right) =: B_1$, $(0,0) = O_{1\times2} = O^T_{2\times1}$,

and represent matrix P as follows:

$$P = \begin{pmatrix} A & O_{2\times2} & O_{2\times1} \\ O_{2\times2} & A & O_{2\times1} \\ B_1 & O_{1\times2} & \frac{1}{2} \end{pmatrix}$$

and use block matrix multiplication (it is similar to standard matrix multiplication, but blocks are treated as " elements"):

$$P^2 = \begin{pmatrix} A & O_{2\times2} & O_{2\times1} \\ O_{2\times2} & A & O_{2\times1} \\ B_1 & O_{1\times2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} A & O_{2\times2} & O_{2\times1} \\ O_{2\times2} & A & O_{2\times1} \\ B_1 & O_{1\times2} & \frac{1}{2} \end{pmatrix} =$$

$$= \begin{pmatrix} A^2 & O_{2\times2} & O_{2\times1} \\ O_{2\times2} & A^2 & O_{2\times1} \\ B_1A+\frac{1}{2}B_1 & O_{1\times2} & \frac{1}{2^2} \end{pmatrix} =$$

[note that

$$A^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = A$$

$$BA+\frac{1}{2}B = \left(\tfrac{1}{4}, \tfrac{1}{4}\right)\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + \frac{1}{2}\left(\tfrac{1}{4}, \tfrac{1}{4}\right) = \left(\tfrac{3}{8}, \tfrac{3}{8}\right)$$]

$$= \begin{pmatrix} A & O_{2\times2} & O_{2\times1} \\ O_{2\times2} & A & O_{2\times1} \\ B_2 & O_{1\times2} & \frac{1}{2^2} \end{pmatrix},$$

where $B_2 = \left(\frac{3}{2^3}, \frac{3}{2^3}\right)$.

By multiplying further, it is easy to see that

$$P^n = \begin{pmatrix} A & O_{2\times2} & O_{2\times1} \\ O_{2\times2} & A & O_{2\times1} \\ B_n & O_{1\times2} & \frac{1}{2^n} \end{pmatrix}, \quad \text{where } B_n = \left(\frac{2^n-1}{2^{n+1}}, \frac{2^n-1}{2^{n+1}}\right)$$

i.e.

$$P^n = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{2^n-1}{2^{n+1}} & \frac{2^n-1}{2^{n+1}} & 0 & 0 & \frac{1}{2^n} \end{pmatrix}$$

Now, for any $i = 0, 1, 2, 3$:

$$\sum_{n=1}^{\infty} P_{i,i}^{n} = \sum_{n=1}^{\infty} \frac{1}{2} = \infty \Rightarrow \text{states } 0, 1, 2, 3 \text{ are recurrent;}$$

for $i = 4$:

$$\sum_{n=1}^{\infty} P_{4,4}^{n} = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 < \infty \Rightarrow \text{state } 4 \text{ is transient.}$$