STK2130, week 2

V21

Ch. 4 (Markov chains), exercises:

4.14, 4.20, 4.23, 4.30

o—o

4.14.) Specify the classes of the following Markov chains & determine whether transient or recurrent:

a) $$P_1 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 2 & \frac{1}{2} & 0 & \frac{1}{2} \\ 3 & \frac{1}{2} & \frac{1}{2} & 0 \end{array}$$

Note that $1 \leftrightarrow 2$ and $2 \leftrightarrow 3$, so all states communicate.

Pos. prob. of going from 1 to 2, 1 to 3, 2 to 1, 2 to 3, 3 to 1 and 3 to 2

Hence, there is only one class in the Markov chain, i.e, the Markov chain is irreducible.

Recurrence and transience are class properties (Cor. 4.2). Not all states in a finite Markov chain can be transient (see comment after Prop. 4.1). Hence, since our Markov chain is irreducible & finite state, all states must be recurrent (see also Remark (ii) right before Ex. 4.17).

b) $$P_2 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 4 & 0 & 0 & 1 & 0 \end{array}$$

$1 \to 4$

$2 \to 4$

$3 \to 1$ or $3 \to 2$

$4 \to 3$

$\Downarrow$

$1 \leftrightarrow 4$, $1 \leftrightarrow 3$, $2 \leftrightarrow 4$, $2 \leftrightarrow 3$ $\Rightarrow$ all states communicate.

That is, we have a finite state, irreducible Markov chain. Hence, all states are recurrent (see ex. a)).

c)
$$P_3 = \begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array}$$

$$\left.\begin{array}{l} 1 \to 1, \quad 1 \to 3 \\ 2 \to 1, \quad 2 \to 2, \quad 2 \to 3 \\ 3 \to 1, \quad 3 \to 3 \end{array}\right\} \quad \begin{array}{c} 3 \leftrightarrow 1 \\ 2 \end{array}$$

$$\left.\begin{array}{l} 4 \to 4, \quad 4 \to 5 \\ 5 \to 4, \quad 5 \to 5 \end{array}\right\} \quad 4 \leftrightarrow 5$$

$$\Downarrow$$

The classes are: $\{2\}$, $\{1,3\}$, $\{4,5\}$

Note that if starting in 2, we will not return (with probability 1). Hence, $\{2\}$ is transient.

The classes $\{1,3\}$ and $\{4,5\}$ are both recurrent, since there is probability 1 of reentering any of these states at some point. F. ex: Starting in 1, what is

$$f_1 = P(\text{reenter } 1 \mid \text{start in } 1) = 1 - P(\text{never reenter } 1 \mid \text{start in } 1)$$

$$= 1 - P(\text{go from } 1 \text{ to } 3 \text{ then always } 3 \mid \text{start in } 1)$$

$$= 1 - \left(\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdots \text{infinitely many times}\right)$$

$$\longrightarrow 1 - 0 = 1$$

d)

$$P_4 = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ 2 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 & 0 \\ 4 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 5 & 1 & 0 & 0 & 0 & 0 \end{array}$$

$$\left.\begin{array}{l} 1 \to 1, \quad 1 \to 2 \\ 2 \to 1, \quad 2 \to 2 \end{array}\right\} \{1, 2\}$$

$3 \to 3 \quad \} \ \{3\}$

$4 \to 3, \quad 4 \to 4 \quad \} \ \{4\}$

$5 \to 1 \quad \} \ \{5\}$

If we start in $\{3\}$, we stay there (with prob. 1). Hence, $\{3\}$ is recurrent.

If we start in 5, we leave (with prob. 1) & never reenter. Hence, $\{5\}$ is transient.

If we start in $\{4\}$, we will leave to 3 with prob. 1 at some point (parallel to argument in c) for reentering 1). Once we leave 4, to 3, we never return. Hence, $\{4\}$ is transient.

Starting in either 1 or 2, there is prob. 1 of reentering the state (see argument in c)). Hence, $\{1, 2\}$ is recurrent.

4.20) A transition prob. matrix is doubly stochastic if the sum of each column is one;

$$\sum_i P_{ij} = 1 \quad \forall j$$

If such a chain is irreducible and consists of $M+1$ states $0, 1, \ldots, M$, show that the long run proportions are:

$$\pi_j = \frac{1}{M+1}, \quad j = 0, 1, \ldots, M$$

oo

The Markov chain is finite state (since # states = $M+1$) and irreducible. Hence, it is positive recurrent (see Remark (ii) pg. 217).

Otherwise, all long run proportions are 0: Not possible for finite # states

Hence, from Thm. 4.1, the long run proportions are the unique solution of:

$$(\star)\quad \begin{cases} \pi_j = \sum_i \pi_i P_{ij}, & j \geq 1 \\ \sum_j \pi_j = 1 \end{cases}$$

Note that for $\pi_j = \frac{1}{M+1}$, $j = 0, \ldots, M$, we have

$$\sum_i \pi_i P_{i,j} = \sum_i \frac{1}{M+1} P_{i,j} = \frac{1}{M+1} \sum_i P_{i,j} = \frac{1}{M+1} = \pi_j$$

doubly stochastic

Hence, $\pi_j = \frac{1}{M+1}$, $j = 0, \ldots, M$

is the unique solution of $(\star)$ for our Markov chain, i.e. these are the long-run proportions.

4.23) Good weather year: # storms $\sim$ Poisson (1)

Bad weather year: # storms $\sim$ Poisson (3)

Suppose: Yearly weather only depend on last year. :(1)

(2): $P(X_n = G \mid X_{n-1} = G) = P(X_n = B \mid X_{n-1} = G) = \frac{1}{2}$

(3): $P(X_n = G \mid X_{n-1} = B) = \frac{1}{3}$

(4): $P(X_n = B \mid X_{n-1} = B) = \frac{2}{3}$ (B twice as likely to be followed by B as G)

$$X_0 = G$$

a) Find $E[\#\text{ storms in } X_1 \text{ \& } X_2 \mid X_0 = G]$?

$\{X_n\}_{n \geq 0}$ is a Markov chain (from (1)) with values in $S = \{G, B\}$ and index set $\{0, 1, 2, \dots\}$, which represents years.

Transition matrix (from (2)-(4)):

$$P = \begin{array}{c} G \\ B \end{array} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$
$$\qquad\quad G \qquad B$$

Define:

$$N_i := \#\text{ storms in year } i.$$

We want to find:

$$E[N_1 + N_2 \mid X_0 = G] \overset{\text{expec. linear}}{=} E[N_1 \mid X_0 = G] + E[N_2 \mid X_0 = G]$$

$$\overset{\text{double expectation rule}}{=} \underbrace{E[E[N_1 \mid X_1] \mid X_0 = G]}_{M1} + \underbrace{E[E[N_2 \mid X_2] \mid X_0 = G]}_{M2} = (\sim)$$

double expectation rule

$E[E[Y \mid X]] = E[Y]$

so:

$E[Y \mid Z] = E[E[Y \mid X] \mid Z]$

First:

M1:
$$E[E[N_1 \mid X_1] \mid X_0 = G]$$

$$\overset{\text{def. expectation}}{=} \sum_{i \in S} E[N_1 \mid X_1 = i] P(X_1 = i \mid X_0 = G)$$

$$\overset{\text{def } S}{=} E[N_1 \mid X_1 = G] P_{G,G} + E[N_1 \mid X_1 = B] P_{G,B}$$

$$= 1 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2} = \frac{1+3}{2} = \underline{2}$$

assumptions; # storms Poisson with means 1 & 3 resp.

+ transition matrix

M2:

Def expectation

$$E[E[N_2 | X_2] | X_0 = G] = \sum_{i \in S} E[N_2 | X_2 = i] \, P(X_2 = i | X_0 = G)$$

def S

$$= E[N_2 | X_2 = G] \, P(X_2 = G | X_0 = G) + E[N_2 | X_2 = B] \, P(X_2 = B | X_0 = G)$$

$$= 1 \cdot P^{(2)}_{G,G} + 3 \cdot P^{(2)}_{G,B}$$

assumptions, ~ Poisson mean 1 & 3 resp.

$$= \frac{5}{12} + \frac{3 \cdot 7}{12} = \frac{26}{12} = \underline{\frac{13}{6}}$$

To find $\underset{\sim}{P^{(2)}_{G,G}}$ & $\underline{P^{(2)}_{G,B}}$:

$$P^2 = \begin{array}{c|cc} & G & B \\ \hline G & \frac{5}{12} & \frac{7}{12} \\ B & \frac{7}{18} & \frac{11}{18} \end{array}$$

Hence; from (~), M1 & M2:

$$E[N_1 + N_2 | X_0 = G] = 2 + \frac{13}{6} = \underline{\underline{\frac{25}{6}}}$$

b) Find $P(\text{no storms year } 3 \mid X_0 = G)$:

$$P(N_3 = 0 \mid X_0 = G) \overset{\text{condition wrt } X_3}{=} \sum_{i \in S} P(N_3 = 0 \mid X_3 = i) P(X_3 = i \mid X_0 = G)$$

$$= P(N_3 = 0 \mid X_3 = G)\, P^{(3)}_{G,G} + P(N_3 = 0 \mid X_3 = B) \cdot P^{(3)}_{G,B}$$

$$P^3 = \begin{array}{c|cc} & G & B \\ \hline G & 0{,}40 & 0{,}60 \\ B & 0{,}40 & 0{,}60 \end{array}$$

$$= e^{-1} \cdot 0{,}4 + e^{-3} \cdot 0{,}6$$

$$\approx \underline{0{,}18}$$

Also:

$P(N_3 = 0 \mid X_3 = G) \sim \text{Poisson}(1)$,

$P(N_3 = 0 \mid X_3 = B) \sim \text{Poisson}(3)$.

Hence:

$$P(N_3 = 0 \mid X_3 = G) = \frac{1^0 e^{-1}}{0!} = e^{-1}$$

$$P(N_3 = 0 \mid X_3 = B) = \frac{3^0 e^{-3}}{0!} = e^{-3}$$

c) Find the long-run average number of storms per year:

"$\underbrace{E[N_\infty}_{} \mid X_0 = G] = ?$"

Write N_∞ although it's a little imprecise. Actually, were interested in $\lim_{n \to \infty} E[N_n \mid X_0 = G]$ but since it's a pain to write over & over, we simplify by writing N_∞

$$E[N_\infty \mid X_0 = G] = E[N_\infty \mid X_\infty = G]\, P(X_\infty = G \mid X_0 = G]$$
$$+ E[N_\infty \mid X_\infty = B]\, P(X_\infty = B \mid X_0 = G)$$

$$= E[N_\infty \mid X_\infty = G]\, \pi_G + E[N_\infty \mid X_\infty = B]\, \pi_B = (\sim$$

def. of long run proportions

$\underline{M}$: Need to find $\vec{\pi} = (\pi_G, \pi_B)$!

Solve
$$\begin{cases} \pi = \pi \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \\ \pi_G + \pi_B = 1 \end{cases}$$

Our Markov chain is a finite state, and irreducible, hence positive recurrent. From Thm 4.1, the long-run proportions are the unique solution of $\pi = \pi P$ & $\sum_i \pi_i = 1$

That is:
$$\pi_G = \frac{1}{2}\pi_G + \frac{1}{3}\pi_B$$
$$\Downarrow$$
$$\pi_B = \frac{3}{2}\pi_G = \frac{3}{2}(1 - \pi_B)$$

$$\pi_B + \frac{3}{2}\pi_B = \frac{3}{2}$$

$$\pi_B\left(1 + \frac{3}{2}\right) = \frac{3}{2}$$

$$\frac{5}{2}\pi_B = \frac{3}{\cancel{2}}$$

$$\pi_B = \underline{\frac{3}{5}}, \text{ so } \pi_G = 1 - \frac{3}{5} = \underline{\frac{2}{5}}$$

Hence, from ($\sim$) and (M):

$$\text{"}E[N_\infty \mid X_0 = G]\text{"} = 1 \cdot \frac{2}{5} + 3 \cdot \frac{3}{5}$$

$X_\infty = G$: $N_\infty \sim \text{Poisson}(1)$

$X_\infty = B$: $N_\infty \sim \text{Poisson}(3)$

$$= \frac{2+9}{5} = \frac{11}{5}$$

4.30) 3 out of 4 trucks are followed by a car.
1 out of 5 cars is followed by a truck.
What fraction of vehicles are trucks?

Make Markov chain of observed vehicles with transition matrix

$$P = \begin{array}{c} C \\ T \\ \end{array} \begin{array}{c} \begin{bmatrix} \frac{4}{5} & \frac{1}{5} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix} \\ \begin{array}{cc} C & \quad T \end{array} \end{array}, \quad S = \{C, T\}$$

Want to find long-run proportion of trucks.

Our Markov chain is finite state and irreducible, hence positive recurrent. Thm. 4.1 tells us that the long-run proportions of cars and trucks, $\pi := (\pi_C, \pi_T)$ is the unique solution of

$$\begin{cases} \pi = \pi P \\ \pi_C + \pi_T = 1 \end{cases}$$

So: $\pi_C = \frac{4}{5}\pi_C + \frac{3}{4}\pi_T$

$$\frac{1}{5}\pi_C = \frac{3}{4}\pi_T$$

$$\pi_C = \frac{15}{4}\pi_T = \frac{15}{4}(1-\pi_C)$$

$$\pi_C = \frac{15}{4} - \frac{15}{4}\pi_C$$

$$\frac{4+15}{\cancel{4}}\pi_C = \frac{15}{\cancel{4}}$$

$$\pi_C = \frac{15}{19}, \text{ so } \pi_T = 1 - \frac{15}{19} = \frac{4}{19}$$

The percentage of trucks on the road is $\pi_T = \frac{4}{19}$.