STK 2130;

Exercises, Spring 2021

# Ch. 5; Exponential distribution & Poisson processes

Today: The Poisson process

Ex: 5.35, 5.37, 5.39, 5.40, 5.41, 5.45

5.35.) $T_1$, time of first event of $\{N(t), t \geq 0\}$, a Poisson process with rate $\lambda$. Let $\lambda_{T_1}(t)$ be a failure rate function. Using

$$P(t < T_1 < t+h \mid T_1 > t) \underset{(\star)}{=} \lambda_{T_1}(t)\, h + o(h),$$

show that $T_1 \sim \exp(\lambda)$.

Follow hint:

$$\underbrace{P(t < T_1 < t+h \mid T_1 > t)} = P(N(t+h) - N(t) > 0 \mid N(t) = 0)$$

Prob. time of first event is between $t$ & $t+h$ when we know no event has happened at $t$

$$= P(N(t+h) - N(t) \geq 0)$$

since Poisson processes have independent increments, the value of $N(t)$ does not impact the distr. of the # events in $(t, t+h)$

$$= P(N(t+h) - N(t) = 1) + P(N(t+h) - N(t) \geq 2)$$

since $\geq 0$, either $=1$ OR 2 or more

$$= \lambda h + o(h) + o(h)$$

Def. 5.2, iii) & iv); Def Poisson proc.

$$= \lambda h + o(h)$$

$o(h) + o(h) = o(h)$

So: From calc. above,

$$P(t < T_1 < t+h \mid T_1 > t) = \lambda h + o(h)$$

But from ($\star$),

$$P(t < T_1 < t+h \mid T_1 > t) = \lambda_{T_1}(t) h + o(h)$$

Hence,

$$\underbrace{\lambda_{T_1}(t)}_{\text{Failure rate function of } T_1} = \lambda$$

From pg. 300, the failure rate function uniquely determines the distribution, and the exp. distribution with rate $\lambda$ is the only distribution with failure rate function constantly equals $\lambda$.

Hence, since $\lambda_{T_1}(t) = \lambda$, $T_1 \sim \exp(\lambda)$.

5.37.) $\{N(t), t \geq 0\}$, Poisson, rate $\lambda$. Let $i \leq n$, $s < t$.

a) $P(N(t) = n \mid N(s) = i) = P(N(t) - N(s) = n-i \mid N(s) = i)$

$$= P(N(t) - N(s) = n-i)$$

Poisson process has independent increments, so $N(s)=i$ does not affect $N(t)-N(s)$

$$= P(N(\overbrace{(t-s)}^{\tilde{t}}+s) - N(s) = n-i)$$

rewrite

$$= e^{-\lambda \tilde{t}} \frac{(\lambda \tilde{t})^{n-i}}{(n-i)!} \quad \left(\text{since } n-i \geq 0,\ t-s > 0\right)$$

(Remark i) after Thm 5.1; $N(\tilde{t}+s) - N(s) \sim \text{Poisson}(\lambda\tilde{t})$)

②

b) $P(N(s)=i \mid N(t)=n)$

$$= \frac{P(N(s)=i,\ N(t)=n)}{P(N(t)=n)}$$

Cond. prob:

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

$$= \frac{P(N(t)=n \mid N(s)=i)\ P(N(s)=i)}{P(N(t)=n)}$$

Change roles of A & B

$$= \frac{e^{-\lambda \tilde{t}} \frac{(\lambda \tilde{t})^{n-i}}{(n-i)!}\ e^{-\lambda s}\frac{(\lambda s)^i}{i!}}{e^{-\lambda t}\frac{(\lambda t)^n}{n!}}$$

a) & Thm. 5.1

since $n\geq 0$, $n-i\geq 0$, $i\geq 0$

$$= \frac{\cancel{e^{-\lambda(t-s)}}\ \cancel{e^{-\lambda s}}\ (\cancel{\lambda}(t-s))^{n-i}\ (\cancel{\lambda} s)^i\ n!}{\cancel{e^{-\lambda t}}\ (\cancel{\lambda} t)^n\ (n-i)!\ i!}$$

$$= \frac{(t-s)^{n-i}\ s^i}{t^n}\binom{n}{i}$$

5.39) Scientific theory: Mistakes in cell division occur as Poisson process, rate 2,5 per year. Individual dies when 196 such mistakes have occurred.

a) Mean lifetime of individual:

Let $N(t) :=$ # mistakes in cell division for person of age $t$ years

By assumption, this is a Poisson process.

Let $T_1 :=$ time of first mistake and

$T_n :=$ time between $(n-1)$st and $n$'th mistake.

"event"

Also, let $S_n :=$ time of $n$'th mistake.

Then,

E[ lifetime of individual] = E [time of 196'th mistake]

$$= E[S_{196}] = \frac{196}{2{,}5} = \underline{\underline{78{,}4}}$$

Comment after Prop. 5.4;

$S_{196} \sim$ gamma$(196, 2.5)$

b) Var [lifetime of individual] = Var [time of 196'th mistake]

$$= Var(S_{196}) = \frac{196}{(2{,}5)^2} = \underline{31{,}36}$$

Approximate;

c) P( individual dies before 67,2)

$$= P(S_{196} \leq 67,2) = F_{S_{196}}(67,2)$$

< or ≤ doesn't matter since S ~ gamma, which is a continuous distribution

where $F_{S_{196}}$ is the cdf of a gamma distribution with parametres 196 & 2,5.

Can look this up in gamma distribution table or via python/ Matlab

d) P( individual becomes 90)

$$= P(S_{196} \geq 90) = 1 - P(S_{196} \leq 90)$$

gamma is a continuous distribution

$$= 1 - F_{S_{196}}(90)$$

e) P(individual becomes 100) = $P(S_{196} \geq 100)$

$$= 1 - P(S_{196} \leq 100)$$

$$= 1 - F_{S_{196}}(100)$$

5.40) $\{N_i(t), t \geq 0\}$ indep. Poisson processes, rate $\lambda_i$, $i=1,2$, then $\{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda_1 + \lambda_2$ where $N(t) = N_1(t) + N_2(t)$.

Def. 5.2

Pf: We check the definition of a Poisson process for $\{N(t), t \geq 0\}$, with rate $\lambda_1 + \lambda_2$:

Def. 5.2:

i) $N(0) \underset{\text{def. } N}{=} N_1(0) + N_2(0) \underset{N_1 \,\&\, N_2 \text{ Poisson proc.}}{=} 0 + 0 = 0.$ ✓

ii) $\{N(t), t \geq 0\}$ has independent increments since $N(t) = N_1(t) + N_2(t)$ and $\{N_1(t), t \geq 0\}$, $\{N_2(t), t \geq 0\}$ are independent and also both Poisson processes, so they have independent increments. ✓

iii) $P(N(t+h) - N(t) = 1) = P(N_1(t+h) + N_2(t+h) - (N_1(t) + N_2(t)) = 1)$

$= P((N_1(t+h) - N_1(t)) + (N_2(t+h) - N_2(t)) = 1)$

either $N_2$ has 1 event & $N_1$ zero, or opposite

$= P(N_1(t+h) - N_1(t) = 1)\, P(N_2(t+h) - N_2(t) = 0)$

$+ P(N_1(t+h) - N_1(t) = 0)\, P(N_2(t+h) - N_2(t) = 1)$

Def 5.2

$$= (\lambda_1 h + o(h))\,(1-\lambda_2 h + o(h) - o(h))$$
$$+ (1-\lambda_1 h - o(h) - o(h))\,(\lambda_2 h + o(h))$$

$$= (\lambda_1 h + o(h))\,(1-\lambda_2 h + o(h))$$
$$+ (1-\lambda_1 h + o(h))\,(\lambda_2 h + o(h))$$

calculation rules for $o(h)$

$$= \lambda_1 h - \lambda_1\lambda_2 h^2 + \lambda_1 h\, o(h)$$
$$+ o(h) - \lambda_2 h\, o(h) + o(h)^2$$
$$+ \lambda_2 h + o(h) - \lambda_1\lambda_2 h^2 - \lambda_1 h\, o(h)$$
$$+ \lambda_2 h\, o(h) + o(h)^2$$

$$= \lambda_1 h + \lambda_2 h + o(h) = (\lambda_1 + \lambda_2)\, h + o(h).$$

$\lambda_i h\, o(h)$ is $o(h)$ & $h^2$ is $o(h)$ because: $\lim_{h\to 0} \frac{h^2}{h} = \lim_{h\to 0} h = 0$

iv) $P(N(t+h) - N(t) \geq 2)$

$$= P((N_1(t+h) - N_1(t)) + (N_2(t+h) - N_2(t)) \geq 2)$$

$$= P(N_1(t+h) - N_1(t) = 1)\, P(N_2(t+h) - N_2(t) = 1)$$
$$+ P(N_1(t+h) - N_1(t) \geq 2) + P(N_2(t+h) - N_2(t) \geq 2),$$

$$= (\lambda_1 h + o(h))(\lambda_2 h + o(h))$$
$$+ o(h) + o(h)$$

since $N_1$ & $N_2$ are Poisson processes, use items iii) & iv) in Def. 5.2

$$= \lambda_1 \lambda_2 h^2 + \lambda_1 h\, o(h) + \lambda_2 h\, o(h) + o(h)^2 + o(h) + o(h)$$

$$= o(h). \checkmark$$

$h^2$ is $o(h)$, $h\,o(h)$ is $o(h)$, sum of $o(h)$'s is $o(h)$, $o(h)^2$ is $o(h)$

So $\{N(t) := N_1(t) + N_2(t),\ t \geq 0\}$ is a Poisson process as well.

5.41) In ex. 40, what is prob. first event from $N(t)$ is from $N_1$?

$$P(N_1(t) = 1 \mid N(t) = 1) = P(N_1(t) = 1 \mid N_1(t) + N_2(t) = 1)$$

$$= P(\min\{T_1^{(1)}, T_1^{(2)}\} = T_1^{(1)})$$

Def:
$T_1^{(1)}$ = time of 1st event $\{N_1(t)\}_{t \geq 0}$
$T_1^{(2)}$ = — " — $\{N_2(t)\}_{t \geq 0}$

$$= \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

see comments after Prop. 5.1

5.45) $\{N(t);\ t \geq 0\}$ Poisson process, rate $\lambda$, independent of r.v. $T \geq 0$ with mean $\mu$, variance $\sigma^2$. Find

a) $\mathrm{Cov}(T, N(T)) \underset{\text{def.}}{=} E[(N(T) - E[N(T)]) \cdot (T - E[T])]$

multiply out & use linearity of expectation

$$= E[N(T) \cdot T] - E[T]E[N(T)] \cancel{- E[T]E[N(T)]} \cancel{+ E[N(T)]\underbrace{E[E[T]]}_{=E[T]}}$$

$$= E[E[N(T) \cdot T \mid T]] - E[T]E[E[N(T) \mid T]]$$

double expectation/tower property; $E[X] = E[E[X|Y]]$

sometimes called law of total expectation

$$= E[T\, E[N(T) \mid T]] - E[T]\, E[\lambda T]$$

$$= E[T \lambda T] - \lambda E[T]^2$$

$N(t)$ is Poisson process with rate $\lambda$, so (given $T$) $N(T)$ is Poisson r.v. with rate $\lambda T$ from Thm. 5.1

$$= \lambda(\underbrace{E[T^2] - E[T]^2}_{= Var(T)})$$

$$= \underline{\underline{\lambda \sigma^2}}$$

b) $\mathrm{Var}(N(T)) = E[\mathrm{Var}(N(T) \mid T)] + \mathrm{Var}(E[N(T) \mid T])$

Law of total variance

$= E(\lambda T) + \mathrm{Var}(\lambda T)$

$= \underline{\lambda \mu + \lambda^2 \sigma^2}$

Given T:

$N(T)$ is Poisson r.v. with rate $\lambda T \Rightarrow$

$\mathrm{Var}(N(T) \mid T) = \lambda T$;

Follows from Thm. 5.1

linearity of expectation,

T has expectation $\mu$.

$\mathrm{Var}(\lambda T) = \lambda^2 \mathrm{Var}(T) = \lambda^2 \sigma^2$