6.1) Population of organisms: male & female.
Any male likely to mate with any female in interval with length h with prob. $\lambda h + o(h)$.
Mating immediately gives one offspring, 50/50 male or female.

$N_1(t) :=$ # males in population at t

$N_2(t) :=$ # females in population at t

Derive parameters, v_i, P_{ij}, of the cont. time Markov chain $\{N_1(t), N_2(t)\}$.

Assume $\{N_1(t), N_2(t)\} = (n_1, n_2)$.

Then,

$$v_{(n_1, n_2)} = \frac{1}{E[T]} \quad \text{where } T \sim \exp(n_1 n_2 \lambda)$$

parameter of exp. distr. denoting the amount of time the state spends in (n_1, n_2) before transitioning to a different state

mean exp. distr. $= \frac{1}{v_i}$

$\Downarrow$

$v_i = \frac{1}{\text{mean exp. distr.}}$

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The amount of time the process spends in the state (n_1, n_2). This is, from the def. of cont. time Markov chains, exponential with rate proportional to the rate of meeting $(n_2 \cdot n_1)$ and the rate of offspring production from mating (λ) ⇝

$$\underbrace{n_1 \cdot n_2}_{\text{rate of mating}} \cdot \underbrace{\lambda}_{\text{rate offspr.} \mid \text{mating}}$$

<u>Note</u>: Reason for this is that only way to leave state (n_1, n_2) is by having mating resulting in offspring

Also, the only states one can go to from (n_1, n_2) is $(n_1 + 1, n_2)$ or $(n_1, n_2 + 1)$ because there is only one offspring per mating, and this is either male or female.

maximally

Hence,

$$P_{(n_1, n_2),(n_1+1, n_2)} = \underline{\underline{P_{(n_1, n_2),(n_1, n_2+1)}}} = \underbrace{\frac{1}{2}}$$

since 50/50 whether offspring is male or female

All other transition probabilities are 0.

6.2) One-celled organism is in one of two states: A or B.
If in A; change to B at exp. rate α.
If in B; divide into $\underline{\text{two}}$ new individuals of type A at exp rate β.

Define a cont. time Markov chain for this model & determine $\underline{\text{parameters}}$.

Let

$$N_A(t) := \# \text{ organisms type A}$$

$$N_B(t) := \# \text{ organisms type B}$$

Now, if $(N_A(t), N_B(t)) = (n_A, n_B)$, then

Alt: $\frac{1}{E[T]}$

$\underbrace{v_{(n_A, n_B)}}$ = rate of T, the exponential time until next event (either $A \to B$ or $B \to A, A$)

rate of exp. distr. denoting time before next event/transition

$$= \underbrace{\alpha n_A} + \beta n_B$$

the n_A type A individuals transfer at rate α

the n_B type B individuals transfer at rate β

The only possible transitions are

1) $(n_A, n_B) \to (n_A - 1, n_B + 1)$

(A transfers to B)

2) $(n_A, n_B) \to (n_A + 2, n_B - 1)$

(B divides into two type A)

All other transition probabilities are 0, and

$P_{(n_A, n_B),(n_A - 1, n_B + 1)}$ = P(A transfers to B before B splits)

$= P(T_A < T_B)$

$T_A :=$ time until type A event $\sim \exp(\alpha n_A)$

$T_B :=$ time until type B event $\sim \exp(\beta n_B)$

Prob. of one exponential (λ_1) r.v. happening before another (λ_2) is:

$$\frac{\lambda_1}{\lambda_1 + \lambda_2}$$

$$= \frac{\alpha n_A}{\alpha n_A + \beta n_B}$$

Similarly,

$P_{(n_A, n_B),(n_A + 2, n_B - 1)}$

= P(B splits before A transfers)

$= P(T_B < T_A)$

$$= \frac{\beta n_B}{\alpha n_A + \beta n_B}$$

Finally, $(N_A(t), N_B(t))$ with these parameters is a continuous time Markov chain since the amount of time spent in a state before transitioning to another is exponentially distributed, and we have transition probabilities between these states.

See alt. def of cont. time Markov chain in beginning of Section 6.2

6.3.) Two machines, maintained by single repairer. Machine i functions for exp. time with rate μ_i before breaking down, $i=1,2$. Repair times: Exp. with rate μ.

Can this be analyzed as a birth/death process? If yes; Parameters? If no; how?

The failure rates, μ_1 and μ_2, are machine dependent (not just depending on e.g., the number of functioning machines). Hence, this does not fit into the framework of birth/death processes in Section 6.3.

However, we can use a continuous time Markov chain to model the system:

State space:

$\star$:= both machines work.

$\dagger_1$:= both machines are down, 1 is in repair.

$\dagger_2$:= —"— —"— , 2 —"—

$\star_1$:= 2 works, 1 down.

$\star_2$:= 1 works, 2 down.

Then,

$$\underbrace{v_\star}_{\substack{\text{rate of}\\ \text{exp. time}\\ \text{spent in}\\ \star \text{ before}\\ \text{transition}}} = \underbrace{\mu_1 + \mu_2}_{\substack{\text{machines 1, 2}\\ \text{break down}\\ \text{with rates}\\ \mu_1,\ \mu_2 \text{ resp.}}} \quad , \quad \underbrace{v_{\dagger_1}} = \underbrace{v_{\dagger_2}} = \mu$$

repair rate for both machines is μ (exp. repair time)

$$\underbrace{v_{\star_1}} = \mu + \mu_2 \quad , \quad \underbrace{v_{\star_2} = \mu + \mu_1}$$

In $\star_1$, 2 works & 1 is down. To transfer from this state, either 1 must be fixed (rate μ) or 2 must break (rate μ_2)

In $\star_2$, 1 works & 2 is down. To move to another state, either 2 must be fixed (rate μ) or 1 must break (rate μ_1).

the transition probabilities are,

$$P_{\star_1,\star_1} = P(1 \text{ fails before } 2) = \frac{\mu_1}{\mu_1+\mu_2}$$

For this transition, 1 must break before 2

lifetime of 1 & 2 are both exp. r.v.'s: The prob. of one exp. (λ_1) failing before $exp(\lambda_2)$ is $\frac{\lambda_1}{\lambda_1+\lambda_2}$

$$P_{\star_1,\star_2} = P(2 \text{ fails before } 1) = \frac{\mu_2}{\mu_1+\mu_2}$$

$$P_{\star_1,+_1} = P(2 \text{ fails before repair of } 1 \text{ is finished}) = \frac{\mu_2}{\mu+\mu_2}$$

Prob. 2 breaks bef. 1 is fixed.

$$P_{\star_2,+_2} = P(1 \text{ fails before repair of } 2 \text{ is finished}) = \frac{\mu_1}{\mu+\mu_1}$$

prob. 1 fails bef. 2 is fixed

$$P_{\star_1,\star} = P(1 \text{ fixed bef. } 2 \text{ fails}) = \frac{\mu}{\mu_2+\mu_1}$$

$$P_{\star_2,\star} = P(2 \text{ fixed bef. } 1 \text{ fails}) = \frac{\mu}{\mu_1+\mu_2}$$

$$P_{+_1,\star_2} = 1 = P_{+_2,\star_1}$$

, since from $+_1$ or $+_2$, the only possible transition is for the machine in repair to be fixed.

6.4.) Potential customers arrive at single server as Poisson process with rate λ. If arrival finds n customers in station, will enter with prob. α_n. Exp. service rate μ. Set up as birth & death process & determine birth / death rates.

Let

$N(t) :=$ # customers in station at time t.

Then, $\{N(t), t \geq 0\}$ is a birth & death process with

$$\underbrace{\lambda_n}_{\substack{\text{birth/}\\\text{arrival}\\\text{rate}\\\text{(exp)}}} = \underbrace{\lambda}_{\substack{\text{rate}\\\text{of}\\\text{arrival}}} \underbrace{\alpha_n}_{\substack{\text{prob. this}\\\text{arrival}\\\text{enters}\\\text{system}\\\text{at state}\\n}}, \quad \overbrace{\mu_n}^{\substack{\text{departure/}\\\text{death rate}}} = \underbrace{\mu}_{\substack{\text{service rate}\\\text{(customers leave}\\\text{as soon as}\\\text{serviced)}}}$$