Ch. 5: The exponential distribution & the Poisson process

STK2130
Spring 21

Today:

Exercises from ch. 5: 5.15, 5.28, 5.30

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5.15.) 100 items simultaneously put on a life test. Suppose the lifetimes of the individual items are $\sim$ exp with mean 200. Test ends at 5 failures; $T := \{\text{end time for test}\}$. What is $E[T]$ and $Var(T)$?

Let

$T_1 :=$ time to first item fails

$T_2 :=$ time from 1st to 2nd failure

$T_3 :=$ —"— 2nd —"— 3rd —"—

$T_4 :=$ —"— 3rd —"— 4th —"—

$T_5 :=$ —"— 4th —"— 5th —"—

Then,

$$T = T_1 + T_2 + T_3 + T_4 + T_5 = \sum_{i=1}^{5} T_i$$

time to 5 items have failed = (time to first failure) + (additional time to 2nd failure) + ... + (additional time to last failure)

Note that,

(lifetime of item 1 ~ exp$(\frac{1}{200})$) (lifetime of item 100 ~ exp$(\frac{1}{200})$)

$$E[T_1] = E[\min\{X_1, \dots, X_{100}\}]$$

$$= \frac{1}{\sum_{i=1}^{100} \frac{1}{200}} = \frac{200}{\sum_{i=1}^{100} 1} = 2 = \frac{200}{100-1+1}$$

(write alteratively; see why soon!)

Recall that

$\min\{X_1, \dots, X_{100}\} \sim \exp\left(\sum_{i=1}^{100} \frac{1}{200}\right)$

from Prop. 5.2.

For simplicity: Assume item 1 fails first, then 2, ... This is w.l.o.g. by renaming the items.

$$E[T_2] = E[\min\{X_2, \dots, X_{100}\}] = \frac{1}{\sum_{i=2}^{100} \frac{1}{200}}$$

(memorylessness; since all are alive at time 1 fails, the exp. distr. starts fresh; no ageing)

(Prop. 5.2 $\min\{X_2, \dots, X_{100}\} \sim \exp\left(\sum_{i=2}^{100} \frac{1}{200}\right)$)

$$= \frac{200}{\sum_{i=2}^{100} 1} = \frac{200}{99} = \frac{200}{100-2+1}$$

Also: $E[T_3] = E[\min\{X_3, \dots, X_{100}\}] = \dots = \frac{200}{100-3+1}$

In general; $E[T_i] = \frac{200}{100-i+1}$ (Check if you're unsure!)

Hence,

$$E[T] = E\left[\sum_{i=1}^{5} T_i\right] = \sum_{i=1}^{5} E[T_i]$$

(def. T)

(linearity of expectation)

$$= \sum_{i=1}^{5} \frac{200}{100-i+1}$$

(above calculations)

Also,

$$Var(T) = Var\left(\sum_{i=1}^{5} T_i\right) = \sum_{i=1}^{5} Var(T_i)$$

(since $T_1, \dots, T_5$ are independent by assumption)

$$= \sum_{i=1}^{5} \left(\frac{200}{100-i+1}\right)^2$$

($X \sim exp(\lambda) \Downarrow Var(X) = \frac{1}{\lambda^2}$)

5.28.) n components with individual lifetimes.
Comp. i functions for exp (λ_i) time.
Assume all comps. are initially used & stay in use until they fail.

a) P(comp. 1 fails second)?

Define X_i := lifetime of comp. $i \sim exp(\lambda_i)$.

(= 0, because 1 can't fail 1st and 2nd)

$$P(\text{comp. 1 fails 2nd}) = P(A \cap \{\min\{X_1, \dots, X_n\} = X_1\})$$

(:= A. Define this as event A, just to not have to write so much)

$$+ P(A \cap \{\min\{X_1, \dots, X_n\} = X_2\})$$

$$+ \dots$$

$$+ P(A \cap \{\min\{X_1, \dots, X_n\} = X_n\})$$

(since one of the X_i's have to fail first, so this constitutes the whole scenario space)

$$= \sum_{i=2}^{n} P(A \cap \{\min\{X_1, \dots, X_n\} = X_i\})$$

(DRAW THIS IN VENN DIAGRAM if unsure)

Cond. prob

$$= \sum_{i=2}^{n} P(A \mid \min\{X_1, \dots, X_n\} = X_i) P(\min\{X_1, \dots, X_n\} = X_i)$$

~~law of total probability~~

$$= \sum_{i=2}^{n} P(\min\{X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n\} = X_1) \times P(\min\{X_1, \dots, X_n\} = X_i)$$

(Know X_i fails first, then $P(X_1 \text{ fails 2nd} \mid X_i \text{ first})$
$= P(\min\{X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n\} = X_1)$)
memoryless

$$= \sum_{i=2}^{n} \frac{\lambda_1}{\sum_{\substack{j \neq i \\ j=1}}^{n} \lambda_j} \cdot \frac{\lambda_i}{\sum_{j=1}^{n} \lambda_j}$$

$\min\{X_1, \dots, X_n\} \sim exp(\sum_{i=1}^{n} \lambda_i)$

From book, right before Prop. 5.2:

$$P(X_i = \min_j X_j) = \frac{\lambda_i}{\sum_{j=1}^{n} \lambda_j}$$

b) E[2nd failure time]?

Define: $T_i :=$ time between failure $(i-1)$ & failure i

Then,

(def. T_i) (linearity of expectation)

$$E[\text{time 2nd failure}] = E[T_1 + T_2] = E[T_1] + E[T_2] = (\sim)$$

M: $E[T_1] = E[\min\{X_1, \dots, X_n\}] = \frac{1}{\sum_{i=1}^{n} \lambda_i}$

Prop. 5.2: $\min\{X_1, \dots, X_n\} \sim exp(\sum_{i=1}^{n} \lambda_i)$

$$E[T_2] = \sum_{i=1}^{n} E[T_2 \mid \min\{X_1, \dots, X_n\} = X_i] \cdot P(\min\{X_1, \dots, X_n\} = X_i)$$

law of total expectation/conditioning

$$= \sum_{i=1}^{n} \frac{1}{\sum_{\substack{j \neq i \\ j=1}}^{n} \lambda_j} \frac{\lambda_i}{\sum_{j=1}^{n} \lambda_j}$$

$$E[T_2 \mid \min\{X_1, \dots, X_n\} = X_i]$$
$$= E[\min\{X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n\}]$$
$$= \frac{1}{\sum_{j \neq i} \lambda_j}$$

<u>Prop. 5.2:</u>

$$\min\{X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n\} \sim \exp\left(\sum_{j \neq i} \lambda_j\right)$$

also, from comment before Prop. 5.2,

$$P(\min\{X_1, \dots, X_n\} = X_i) = \frac{\lambda_i}{\sum_{j=1}^{n} \lambda_j}$$

So, from (~)

$$E[\text{time 2nd failure}] = E[T_1] + E[T_2]$$

$$= \frac{1}{\sum_{i=1}^{n} \lambda_i} + \sum_{i=1}^{n} \frac{1}{\sum_{j \neq i} \lambda_j} \frac{\lambda_i}{\sum_{j=1}^{n} \lambda_j}$$

5.30.) Lifetimes of A's dog & cat $\sim \exp(\lambda_d)$ and $\sim \exp(\lambda_c)$, independent. One of them just died. What is E[lifetime other pet]?

Define:

T := additional lifetime of other pet

$$E[T] = E[T \mid \text{dog died}]\, P(\text{dog died before cat}) + E[T \mid \text{cat died}]\, P(\text{cat died bef. dog})$$

(condition on which pet has died (law of total expectation))

$$= \frac{1}{\lambda_c} \frac{\lambda_d}{\lambda_c+\lambda_d} + \frac{1}{\lambda_d} \frac{\lambda_c}{\lambda_c+\lambda_d}$$

$E[T \mid \text{cat died}]$

$= E[\text{remaining lifetime dog}]$ (dog alive)

$= E[\text{lifetime dog}] = \frac{1}{\lambda_d}$ (memoryless)

Similarly; $E[T \mid \text{dog died}] = \frac{1}{\lambda_c}$

Also:

P(cat died bef. dog)

$= P(\min\{X_c, X_d\} = X_c)$ ($X_c \sim \exp(\lambda_c)$, $X_d \sim \exp(\lambda_d)$)

$= \frac{\lambda_c}{\lambda_c+\lambda_d}$ (comment bef. Prop. 5.2)

Similar for P(dog died bef. cat)