Exercise session 4

Ch. 4, Markov chains:

4.64, 4.65, 4.66

4.64) Branching process, $\mu < 1$. Show: If $X_0 = 1$, the expected number of individuals in this population is $\frac{1}{1-\mu}$.

What if $X_0 = n$?

If $X_0 = 1$, we know from pg. 248 that

$$E[X_n] = \mu^n, \quad n \geq 1.$$

Recall: Found by conditioning on X_{n-1} & using that μ is expected # offspring p.p. gen. $n-1$

Hence

E[# individuals to ever exist]

$$= E[\sum_{n \geq 0} \underbrace{X_n}_{\text{\# individuals generation } n}]$$

$$= \sum_{n\geq 0} E[X_n]$$

linearity of expectation

$$= \sum_{n\geq 0} \mu^n = \frac{1}{1-\mu}$$

above, or pg. 248 for full calculation

Geometric series; $\sum_{n\geq 0} a\mu^n = \frac{a}{1-\mu}, |\mu|<1$. here $a=1$ and $0\leq\mu<1$, so $|\mu|<1$

$$\mu = \sum_{j=0}^{\infty} j P_j$$

So the expected number of individuals in the population is $\frac{1}{1-\mu}$ (for $X_0 = 1$).

If $X_0 = n$, then from the calculation on pg. 248, we find:

$$E[X_1] = \mu E[X_0] = \mu n$$

$$E[X_2] = \mu E[X_1] = \mu\mu n = \mu^2 n$$

$$\vdots$$

$$E[X_m] = \mu^m n$$

Hence,

$$E[\#\text{ individuals to ever exist} \mid X_0 = n]$$

$$= E\Big[\sum_{m \geq 0} X_m\Big]$$

= sum of all generations

$$= \sum_{m \geq 0} E[X_m]$$

linearity of expec.

$$= \sum_{m \geq 0} n\mu^m = \frac{n}{1-\mu}$$

as before: geometric series

So if $X_0 = n$, then the expected number of individuals in the population is $\frac{n}{1-\mu}$.

4.65.) Branching process, $X_0 = 1$, $\mu > 1$. Prove that π_0 is the smallest positive number such that

$$\pi_0 = \sum_{j=0}^{\infty} \pi_0^{\,j} P_j$$

Hint: Use induction to show...

non-negative — since $\pi_0 := P(\text{pop. dies out}) \geq 0$

Let π be any solution to $\pi = \sum_{j=0}^{\infty} \pi^j P_j$.

Want to use induction to show that

$$\pi \geq P(X_n = 0) \ \forall n.$$

$P(X_n = 0 \mid X_0 = 1)$ $\parallel$

Induction hypothesis: $\pi \geq P(X_n = 0)$ for $n = 0, \dots, k$

Check for $k = 0$:

since π is non-neg

$$P(X_0 = 0) = 0 \leq \pi$$

By assumption, $X_0 = 1$

Induction step:

condition on X_1

$$P(X_{k+1} = 0) = \sum_{j \geq 0} P(X_{k+1} = 0 \mid X_1 = j)\, P(X_1 = j \mid X_0 = 1)$$

$\mid X_0 = 1$

from independence

$$= \sum_{j \geq 0} [P(X_{k+1} = 0 \mid X_1 = 1)]^j\, P(X_1 = j \mid X_0 = 1)$$

Prob. of j families not having offspring is

prob. fam. 1 NO $\cdot$ prob. fam. 2 NO $\cdots$ prob. fam. j NO

$= [P(X_{k+1} = 0 \mid X_1 = 1)]^j$

assumed same prob.

Def. P_j

$$= \sum_{j \geq 0} [P(X_{k+1} = 0 \mid X_1 = 1)]^j\, P_j$$

Prob. of N-O after k steps: Shift.

$$= \sum_{j \geq 0} [P(X_k = 0 \mid X_0 = 1)]^j\, P_j \ \} \geq 0$$

$$\leq \sum_{j \geq 0} \pi^j P_j = \pi$$

induction ~~hyp~~ ~~step~~

assumption

So: $P(X_{k+1} = 0) \leq \pi$

Hence, the induction is completed & we know that

" $P(X_n = 0 \mid X_0 = 1)$ "

$$P(X_n = 0) \leq \pi \quad \forall n. \qquad (\star)$$

Now, we pass to the limit; Know that

$$\pi_0 := \lim_{n \to \infty} P(X_n = 0 \mid X_0 = 1)$$

Hence;

$$\pi_0 = \lim_{n \to \infty} P(X_n = 0 \mid X_0 = 1) \leq \pi$$

$(\star)$; inequality preserved in the limit

So $\pi_0 \leq \pi$ for any solution, π, of

$$\pi = \sum_{j \geq 0} \pi^j P_j \qquad (\star\star)$$

This proves that π_0 is the smallest solution of (★★), since we know (from pg. 250) that π_0 actually solves (★★).

4.66) For a branching process, calculate π_0 when

a) $P_0 = \frac{1}{4}$, $P_2 = \frac{3}{4}$

$\mu := \sum_{j \geq 0} j P_j$ expec. # offspring

From 4.65), we know that if $\mu > 1$, then π_0 is the smallest positive number s.t.

$$\pi_0 = \sum_{j \geq 0} \pi_0^j P_j \quad (\star)$$

Check what μ is:

$$\mu = \sum_{j \geq 0} j P_j = 0 \cdot \frac{1}{4} + 1 \cdot 0 + 2 \cdot \frac{3}{4}$$

$$= \frac{6}{4} = \frac{3}{2} > 1,$$ so π_0 is the smallest pos. number solving ($\star$):

$$\pi_0 = \sum_{j \geq 0} \pi_0^j P_j$$

$$= \frac{1}{4} \pi_0^0 + \frac{3}{4} \pi_0^2$$

$$= \frac{1}{4} + \frac{3}{4} \pi_0^2$$

$$\frac{3}{4} \pi_0^2 - \pi_0 + \frac{1}{4} = 0$$

$$\pi_0 = \frac{1 \pm \sqrt{(-1)^2 - 4 \cdot \frac{3}{4} \cdot \frac{1}{4}}}{2 \cdot \frac{3}{4}}$$

$$= \frac{1 \pm \sqrt{1 - \frac{3}{4}}}{\frac{3}{2}} = \frac{2}{3}\left(1 \pm \sqrt{\tfrac{1}{4}}\right)$$

$$= \frac{2}{3}\left(1 \pm \frac{1}{2}\right) = \begin{cases} \frac{2}{3} \cdot \frac{3}{2} \\ \frac{2}{3} \cdot \frac{1}{2} \end{cases} = \begin{cases} 1 \\ \frac{1}{3} \end{cases}$$

Hence, $\pi_0 = \frac{1}{3}$ or $\pi_0 = 1$. Since we know π_0 is the <u>smallest</u> positive number satisfying (*),

$$\underline{\underline{\pi_0 = \frac{1}{3}}}$$

b) $P_0 = \frac{1}{4}$, $P_1 = \frac{1}{2}$, $P_2 = \frac{1}{4}$

Again, from 4.65) we know that if $\mu > 1$, then π_0 is the smallest positive number s.t.

$$\pi_0 = \sum_{j \geq 0} \pi_0^j P_j$$

$$\mu = \sum_{j \geq 0} j P_j = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4}$$

$$= \frac{1}{2} + \frac{1}{2} = 1.$$

Since $\mu = 1$, we know from pg. 249 (bottom) that $\pi_0 = 1$. Hence, the probability of extinction/ the population dying out in this case is 1.

Book pg. 249

NOTE: $\mu \leq 1 \Rightarrow \pi_0 = 1$

$\mu > 1 \Rightarrow \pi_0$ is smallest, positive solution to $\pi_0 = \sum_{j \geq 0} \pi_0^j P_j$

(Ex. 4.65)

c) $P_0 = \frac{1}{6}$, $P_1 = \frac{1}{2}$, $P_3 = \frac{1}{3}$

Check μ;

$$\mu = \sum_{j \geq 0} j P_j = 0 \cdot \frac{1}{6} + 1 \cdot \frac{1}{2} + 2 \cdot 0 + 3 \cdot \frac{1}{3}$$

$$= \frac{1}{2} + 1 = \frac{3}{2} > 1$$

Hence, from 4.65), π_0 is the smallest, positive solution of:

$$\pi_0 = \sum_{j \geq 0} \pi_0^j P_j$$

I.e.,

$$\pi_0 = \pi_0^0 \frac{1}{6} + \pi_0^1 \frac{1}{2} + \pi_0^2 \cdot 0 + \pi_0^3 \frac{1}{3}$$

$$= \frac{1}{6} + \frac{1}{2}\pi_0 + \frac{1}{3}\pi_0^3$$

$$\frac{1}{3}\pi_0^3 - \frac{1}{2}\pi_0 + \frac{1}{6} = 0$$

Note that $\pi_0 = 1$ is a solution;

$$\frac{1}{3} - \frac{1}{2} + \frac{1}{6} = \frac{2-3+1}{6} = 0$$

Need to do polynomial division to factor;

$$\frac{1}{3}\pi_0^3 - \frac{1}{2}\pi_0 + \frac{1}{6} \div \pi_0 - 1 = \frac{1}{3}\pi_0^2 + \frac{1}{3}\pi_0 - \frac{1}{6}$$

$$-\left(\frac{1}{3}\pi_0^3 - \frac{1}{3}\pi_0^2\right)$$

$$\frac{1}{3}\pi_0^2 - \frac{1}{2}\pi_0 + \frac{1}{6}$$

$$-\left(\frac{1}{3}\pi_0^2 - \frac{1}{3}\pi_0\right)$$

$$-\frac{1}{6}\pi_0 + \frac{1}{6}$$

$$-\left(-\frac{1}{6}\pi_0 + \frac{1}{6}\right)$$

$$0$$

$$-\frac{1}{2} + \frac{1}{3} = \frac{-3+2}{6} = -1/6$$

Now, can solve $\frac{1}{3}\pi_0^2 + \frac{1}{3}\pi_0 - \frac{1}{6} = 0$

$$2\pi_0^2 + 2\pi_0 - 1 = 0$$

$$\pi_0 = \frac{-2 \pm \sqrt{4 - 4\cdot 2\cdot(-1)}}{4} = \frac{-2 \pm \sqrt{4+8}}{4} = \frac{-2 \pm 2\sqrt{3}}{4}$$

$$= \frac{-1 \pm \sqrt{3}}{2}$$

The only positive solution here is

$$\pi_0 = \frac{-1 + \sqrt{3}}{2} \approx 0{,}37 < 1$$

Hence, $\underline{\underline{\pi_0 = \frac{-1 + \sqrt{3}}{2}}}$.