4.69) M balls dist. among m urns.

- Stage i: Select one ball, place at random in one of $m-1$ other urns.

- $n_i :=$ # balls in urn i

- Markov chain with state: $(n_1, \dots, n_m)$

Guess limiting prob. & verify. Show that the Markov chain is time-reversible.

Guess for limiting probabilities:

State space S is such that

$$i \in S \iff i = (n_1^i, n_2^i, \dots, n_m^i)$$

$$\text{where } \sum_{k=1}^{m} n_k^i = M$$

(★★) ← Ways of choosing n_1^i & ... & n_m^i out of M; unordered

Guess that

$$\underbrace{\pi_i}_{\text{proportion of time spent in state } i\text{; with } n_1^i \text{ balls in urn } 1, \ldots, n_m^i \text{ balls in urn } m.} = \frac{M!}{(n_1^i)! \cdot \ldots \cdot (n_m^i)!} \underbrace{\left[\frac{1}{m}\right]^{n_1^i}}_{\text{prob. of choosing urn 1 for } n_1^i \text{ balls}} \cdot \ldots \cdot \underbrace{\left[\frac{1}{m}\right]^{n_m^i}}_{\text{prob. of choosing urn } m \text{ for } n_m^i \text{ balls}}$$

Probability of having n_1^i balls in 1, ..., n_m^i balls in m

$$= \frac{M!}{(n_1^i)! \cdot \ldots \cdot (n_m^i)!} \left[\frac{1}{m}\right]^M$$

Since $\sum_{j=1}^{m} n_j^i = M$

M; (★★) ⇝ Why is this true?

$$\underbrace{\pi_i}_{\text{guess ?}} = \underbrace{\left(\begin{array}{c}\#\text{ways to choose} \\ n_1^i \text{ balls in } 1, \\ \ldots\ n_m^i \text{ balls in } m\end{array}\right)}_{= \#\text{ of ways to group } M \text{ into } m \text{ groups of sizes } n_1^i, \ldots, n_m^i} \cdot \left(\begin{array}{c}\text{prob. of} \\ \text{choosing} \\ \text{urn } 1 \\ \text{for the} \\ n_1^i \text{ balls}\end{array}\right) \cdot \ldots \cdot \left(\begin{array}{c}\text{prob. of} \\ \text{choosing} \\ \text{urn } m \\ \text{for } n_m^i \\ \text{balls}\end{array}\right)$$

$$= \frac{M!}{(n_1^i)! \cdots (n_m^i)!} \quad (\sim)$$

because e.g.,

ways to group $M = 10$ into $m = 3$ groups of sizes $n_1 = 2$, $n_2 = 5$, $n_3 = 3$ is

$$\underbrace{\frac{10!}{2!\,\cancel{8!}}}_{\text{\# ways to choose } n_1 = 2 \text{ out of } 10\text{; binomial coefficient/combination formula}} \cdot \underbrace{\frac{\cancel{8!}}{5!\;3!}}_{\text{\# ways to choose } n_2 = 5 \text{ out of the remaining } 10-2=8 \text{ balls}} \cdot \underbrace{1}_{\text{no need to choose the final group since we are to choose 3 \& 3 are left}} = \frac{10!}{2!\;5!\;3!}$$

This generalizes (via induction) to $(\sim)$.

We are told to check time-reversibility of the Markov chain & at the same time verify that our guess of limiting probabilities is actually correct:

To check time reversibility, we need to check:

$$\pi_i P_{ij} \underset{?}{=} \pi_j P_{ji} \quad \forall\, i,j$$

Note that $P_{ij} \neq 0 \iff \exists\, r, s \in \{1, \dots, m\}$ s.t.

$$i = (n_1, \dots, n_r, \dots, n_s, \dots, n_m) \;\&$$

$$j = (n_1, \dots, n_r - 1, \dots, n_s + 1, \dots, n_m)$$

That is, $P_{ij} \neq 0$ iff. you take a ball from one urn (r here) and put it in another urn (s here)

$$\Downarrow$$

$$\underbrace{P_{ij}}_{\substack{\text{prob. go}\\ \text{from } i \text{ to}\\ j}} = \underbrace{\frac{n_r}{M}}_{\substack{\text{prob.}\\ \text{choose}\\ \text{one of}\\ \text{the } n_r\\ \text{balls in}\\ r \text{ out of}\\ M \text{ in total}}} \cdot \underbrace{\frac{1}{m-1}}_{\substack{\text{prob.}\\ \text{choose}\\ s \text{ out of}\\ \text{remaining}\\ m-1\\ \text{urns}}} \quad \& \quad \underbrace{P_{j,i}}_{\substack{\text{prob. go}\\ \text{from } j\\ \text{to } i}} = \underbrace{\frac{n_s+1}{M}}_{\substack{\text{prob.}\\ \text{choose}\\ \text{one of}\\ \text{the } n_s+1\\ \text{balls in}\\ s \text{ out of}\\ M \text{ in total}}} \underbrace{\frac{1}{m-1}}_{\substack{\text{choose}\\ r \text{ out}\\ \text{of}\\ m-1\\ \text{urns}}}$$

Therefore,

$$\pi_i P_{ij} = \pi_j P_{ji} \iff$$

$$\frac{n_r}{\cancel{M}} \frac{1}{\cancel{m-1}} \frac{\cancel{M!}}{\cancel{(n_1)!} \cdots \cancel{(n_m)!}} = \frac{n_s+1}{\cancel{M}} \frac{1}{\cancel{m-1}} \frac{\cancel{M!}}{\cancel{(n_1)!} \cdots (n_r-1)! \cdots (n_s+1)! \cdots \cancel{(n_m)!}}$$

$$\Updownarrow$$

$$\frac{n_r}{(n_s)!\,(n_r)!} = \frac{n_s+1}{(n_s+1)!\,(n_r-1)!}$$

On R.H.S. we have $(n_s+1)!$ & $(n_r-1)!$ instead of $(n_s)!$ & $(n_r)!$

$$\Updownarrow$$

$$\frac{1}{(n_s)!\,(n_r-1)!} = \frac{1}{(n_s)!\,(n_r-1)!}$$

Def of faculty

Which is clearly true.

Hence, by tracing the iff. backwards, we find that

$$\pi_i P_{ij} = \pi_j P_{ji} \quad \forall\, i, j, \quad (\star)$$

and hence we have time-reversibility.

From top of pg. 252: If we can find non-negative numbers summing to 1 s.t. $(\star)$ holds, these are the limiting probabilities.

We know $(\star)$ holds, also

$$\pi_i = \underbrace{\frac{M!}{(n_1^i)! \cdots (n_m^i)!}}_{\geq 0} \underbrace{\left(\frac{1}{m}\right)^M}_{\geq 0} \geq 0$$

Finally, if the numbers do not sum to one, we can make this OK by normalizing ((★) & ≥ 0 still holds).

4.73) • k players, $v_i > 0$, $i = 1, \dots, k$ is value of player i.

• Each period: 2 play a game.

• Winner plays against randomly chosen player among $k-1$ remaining.

• If i & j play, i wins with prob. $\dfrac{v_i}{v_i + v_j}$

• X_n = winner game n.

oo

a) Transition prob. of Markov chain $\{X_n\}_{n \geq 1}$;

$\underbrace{P_{i,j}}_{\substack{\text{prob. } i \text{ won} \\ \text{game, then} \\ \text{plays against} \\ j \text{ \& } j \text{ wins} \\ = \text{prob. go from} \\ \text{state } i \text{ to } j}}$ $= P(j \text{ is chosen}, j \text{ wins against } i \mid i \text{ won})$

$= P(\text{choose player } j \text{ out of } k-1) \cdot P(j \text{ wins over } i)$

$= \dfrac{1}{k-1} \cdot \dfrac{v_j}{v_i + v_j}$

Note: This is for i different from j. For i=j, one must sum over all n-1 potential other opponents in game nr. 2.

b) Stationarity equations:

$$\begin{cases} \sum_{i=1}^{k} \pi_i = 1 \\ \vec{\pi} = \vec{\pi} \underbrace{\mathbb{P}}_{\text{transition probability matrix}} \end{cases}$$

Note: $\{X_n\}_{n \geq 1}$ is a finite state, irreducible Markov chain (since $v_i > 0 \ \forall i$, this means that $\frac{v_i}{v_j + v_i} > 0 \ \forall i, j$, so any player can beat any other; Hence, all states communicate) Hence, the stationarity eq. are given as above.

c) Time reversibility equations:

$$\pi_i P_{ij} = \pi_j P_{ji} \quad \forall i, j$$

If also $\pi_i \geq 0 \ \forall i$ & sum to 1, then from pg. 252 we know that these are actually the limiting probabilities.

d) Proportion of games won by j:

This is the same as finding π_j.

Can either solve stationarity eqns. from b) or time-reversibility eqns. from c).

Try to use stationarity eqns: Hopeless!

Instead: Aim to use c) & guess for solution.

$$\pi_i \frac{1}{\cancel{k-1}} \frac{v_j}{\cancel{v_i+v_j}} = \pi_j \frac{1}{\cancel{k-1}} \frac{v_i}{\cancel{v_i+v_j}} \quad \forall\ i \neq j$$

Note that the time reversibility is trivially true for i=j.

$$(\star) \qquad \pi_i v_j = \pi_j v_i \qquad \forall\ i \neq j$$

Guess: $\pi_i = \frac{v_i}{\sum_{j=1}^{k} v_i} > 0$ (since $v_i > 0\ \forall\ i$)

Also, $(\star)$ holds $\forall\ i, j$.

Finally, $\sum_{i=1}^{k} \pi_i = \frac{1}{\sum_{i=1}^{k} v_i} \cancel{\sum_{i=1}^{k} v_i} = 1$

Hence, from pg. 252, this $\vec{\pi}$ is the limiting distribution

so $\pi_j = \frac{v_j}{\sum_{i=1}^{k} v_i}$.

e) Proportion of games involving j as player

$\{\vec{X}_j\}_{j\geq 1}$ is the Markov chain where $\vec{X}_j = (X_j, Y_j)$,
& Y_j is the loser of game j.

Then,

$$\underbrace{P_{(i,j),(w,l)}}_{\text{prob. go to } w \text{ winner } l \text{ loser given } i \text{ won over } j} = \begin{cases} \frac{1}{k-1} \frac{v_i}{v_i + v_l} & \text{, if } w = i,\ l \neq i \quad \text{(same as a)} \\ \frac{1}{k-1} \frac{v_w}{v_i + v_w} & \text{, if } l = i,\ w \neq i \\ 0 & \text{otherwise} \end{cases}$$

$$\# \text{ states} = \underbrace{k}_{\substack{k \text{ people} \\ \text{to choose} \\ \text{from as} \\ \text{winner}}} \cdot \underbrace{(k-1)}_{\substack{\text{one of} \\ \text{remaining} \\ k-1 \text{ is the} \\ \text{loser}}}$$

$$\cancel{\sum_{i=1} \sum_{j \neq i} v_i = \sum_{i=1} v_i (k-1) = (k-1) \sum v_i}$$

Time-reversibility:

$$\pi_{i,j} \, P_{(i,j),(w,l)} = \pi_{w,l} \, P_{(w,l),(i,j)} \quad \forall (w,l), (i,j)$$

$w \neq i$:
$$\pi_{i,j} \cancel{\frac{1}{k-1}} \frac{v_w}{\cancel{v_i + v_w}} = \pi_{w,l} \frac{v_i}{\cancel{v_i + v_w}} \cancel{\frac{1}{k-1}}$$

$$\pi_{i,j} \, v_w = \pi_{w,l} \, v_i$$

Guess:
$$\pi_{i,j} = \frac{v_i}{\sum_{i=1}^{k} \underbrace{\sum_{j \neq i}}_{(k-1 \text{ terms})} v_i} = \frac{v_i}{(k-1) \sum_{i=1}^{k} v_i}$$

Can check, as in d), that this satisfies time-reversibility & is ≥ 0 & sums to 1.

$\pi_{i,j}$ is the proportion of time where i wins against j.

Hence, the proportion of time j loses is

$$\sum_{i \neq j} \pi_{i,j} = \sum_{i \neq j} \frac{v_i}{(k-1) \sum_{i=1}^{k} v_i}$$

$$= \frac{\sum_{i \neq j} v_i}{(k-1) \sum_{i=1}^{k} v_i}$$

Hence, the proportion games j is in is the sum of the time she wins & the times she loses, so

proportion of games j is in $= \dfrac{v_j}{\sum_{i=1}^{k} v_i} + \dfrac{\sum_{i \neq j} v_i}{(k-1) \sum_{i=1}^{k} v_i}$

From d) and above

$$= \frac{v_j (k-1) + \sum_{i \neq j} v_i}{(k-1) \sum_{i=1}^{k} v_i}$$

Exercise 2

a

$$P = \begin{bmatrix} 0 & p & 1-p \\ 1-p & 0 & p \\ p & 1-p & 0 \end{bmatrix}$$

b The chain is ergodic because all states communicate and p cannot assume values 0 and 1. By symmetry, $\pi_1 = 1/3, \pi_2 = 1/3, \pi_3 = 1/3$.

c

$$\begin{aligned} P[X_n = 1, X_{n+1} = 2] &= P[X_{n+1} = 2 | X_n = 1] P[X_n = 1] \\ &= p_{12} P[X_n = 1]. \end{aligned}$$

For n large, $P[X_n = 1] = \pi_1 = 1/3$, therefore $P[X_n = 1, X_{n+1} = 2] = p/3$. Similarly, $P[X_n = 1, X_{n+1} = 2] = (1-p)/3$.

d To be time reversible, $\pi_i p_{ij} = \pi_j p_{ji}$ should hold. Since in this case $\pi_i = \pi_j \ \forall i, j$ then it must hold $p_{ij} = p_{ji}$, i.e., $p = 1 - p$, so $p = 1/2$.

NB Note that p and $1 - p$ may be inverted in the solution if the triangle was differently oriented.