1. Definition of Poisson process

Definition A stochastic process $\{N(t), t \geq 0\}$ is said to be a counting process if $N(t)$ represents the total number of events that occur by time $t \geq 0$.

- $N(t) \in \{0, 1, 2, \dots\}$ for all $t \geq 0$;
- $N(t)$ is non-decreasing in t, i.e. $N(s) \leq N(t)$ if $s < t$;
- If $s < t$, then $N(t) - N(s)$ equals the number of events in $(s, t]$.

Example

1) $N(t) =$ "number of people arriving at a store up to and including time $t \geq 0$";
2) $N(t) =$ "Number of children being born up to and including time $t \geq 0$";
3) $N(t) =$ "Number of reported accidents up to and including time $t \geq 0$".

Definition A counting process is said to have independent increments if, for each $s_1 < t_1 \leq s_2 < t_2$ random variables $N(t_1) - N(s_1)$ and $N(t_2) - N(s_2)$ are independent.

Definition A counting process is said to have stationary increments, if for a fixed t, the distribution of $N(s+t) - N(s)$ is the same for all $s \geq 0$.

Definition The function $f(\cdot)$ is said to be $o(h)$ as $h \to 0$ if
$$\lim_{h\to 0} \frac{f(h)}{h} = 0.$$

Example

1) $f(h) = h^2$: $\quad \lim_{h\to 0} \frac{h^2}{h} = \lim_{h\to 0} h = 0 \implies h^2 = o(h),\ h \to 0.$

2) $f(h) = h$: $\quad \lim_{h\to 0} \frac{h}{h} = 1 \implies h$ is not $o(h),\ h \to 0.$

3) If f and g are $o(h), h \to 0$, then $f+g$ is also $o(h),\ h \to 0.$

4) If f is $o(h), h\to 0$, then $c \cdot f$ is also $o(h),\ h \to 0.$

5) Any finite linear combination of $o(h), h \to 0$, functions is also $o(h), h \to 0.$

Definition The counting process $\{N(t), t \geq 0\}$ is said to be a (standard) Poisson process with rate $\lambda > 0$, if

(i) $N(0) = 0$;

(ii) $\{N(t), t \geq 0\}$ has independent increments;

(iii) $\mathbb{P}(N(t+h) - N(t) = 1) = \lambda h + o(h),\ h \to 0;$

(iv) $\mathbb{P}(N(t+h) - N(t) \geq 2) = o(h),\ h \to 0.$

2. Properties of Poisson process

Lemma 5.1 Assume that $\{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda > 0$. Then let $s > 0$ and define
$$N_s(t) = N(s+t) - N(s),\ t \geq 0.$$
Then $\{N_s(t), t \geq 0\}$ is a Poisson process with rate $\lambda > 0$ as well.

Proof We will simply check the definition.

(i) $N_s(0) = N(t) - N(t) = 0$;

(ii) $\{N_s(t), t \geq 0\}$ has independent increments since $\{N(t), t \geq 0\}$ has independent increments;

(iii) $\mathbb{P}(N_s(t+h) - N_s(t) = 1) = \mathbb{P}(N(s+t+h) - N(s+t) = 1) = \lambda h + o(h)$, $h \to 0$;

(iv) $\mathbb{P}(N_s(t+h) - N_s(t) \geq 2) = \mathbb{P}(N(s+t+h) - N(s+t) \geq 2) = o(h)$, $h \to 0$. ◢

Lemma 5.2 If T_1 is the time of the first event of the Poisson process, i.e. $T_1 = \inf\{t \geq 0 \mid N(t) = 1\}$, then

$$\mathbb{P}(T_1 > t) = \mathbb{P}(N(t) = 0) = e^{-\lambda t}, \quad t \geq 0.$$

That is, T_1 has exponential distribution.

Proof Let $P_0(t) := \mathbb{P}(N(t) = 0)$. Then

$$P_0(t+h) = \mathbb{P}(N(t+h) = 0) = \left(\begin{array}{l} N \text{ is a counting process} \\ \Rightarrow \text{ it is non-decreasing} \end{array}\right) =$$

$$= \mathbb{P}(\{N(t) = 0\} \cap \{N(t+h) - N(t) = 0\}) = (N(0) = 0)$$

$$= \mathbb{P}(\{N(t) - N(0) = 0\} \cap \{N(t+h) - N(t) = 0\}) = \left(\begin{array}{l} \text{independent} \\ \text{increments} \end{array}\right)$$

$$= \mathbb{P}(N(t) = 0) \cdot \underbrace{\mathbb{P}(N(t+h) - N(t) = 0)}_{\substack{1 - \mathbb{P}(N(t+h) - N(t) = 1) - \mathbb{P}(N(t+h) - N(t) \geq 2) \\ = 1 - \lambda h + o(h)}}$$

$$= P_0(t)(1 - \lambda h + o(h)), \quad h \to 0.$$

Hence we get:

$$P_0(t+h) - P_0(t) = -\lambda h P_0(t) + o(h) P_0(t), \quad h \to 0.$$

Dividing by h and letting $h \to 0$ we obtain:

$$P_0'(t) = -\lambda P_0(t), \quad P_0(0) = 1 \text{ (since } N(0) = 0).$$

Thus

$$\frac{P_0'(t)}{P_0(t)} = -\lambda \implies \int_0^t \frac{P_0'(s)}{P_0(s)}\,ds = -\lambda t + C$$

$$\implies \log(P_0(t)) = -\lambda t + C.$$

Taking into account that $P_0(0) = 1$, we finally get

$$P_0(t) = e^{-\lambda t}.$$

Proposition 5.4 Let $T_1, T_2, \dots,$ be the sequence of interarrival times of a Poisson process $\{N(t), t \geq 0\}$ with rate $\lambda > 0$. That is, T_1 is the time of the first event, T_n = the time between the $(n-1)$-st and n-th event. Then $T_1, T_2, \dots$ are independent and identically distributed exponential random variables with rate λ.

Proof We have already shown that $T_1 \sim \exp(\lambda)$. For T_2 we get:

$$\mathbb{P}(T_2 > t \mid T_1 = s) = \frac{\mathbb{P}(T_2 > t, T_1 = s)}{\mathbb{P}(T_1 = s)} = \frac{\mathbb{P}(N(t+T_1) - N(T_1) = 0, \ T_1 = s)}{\mathbb{P}(T_1 = s)} =$$

$$= \frac{\mathbb{P}(N(t+s) - N(s) = 0, \ T_1 = s)}{\mathbb{P}(T_1 = s)} = \text{(independent increments)} =$$

$$= \mathbb{P}(N(t+s) - N(s) = 0) = \mathbb{P}(N_s(t) = 0) = e^{-\lambda t} \ \left(\begin{array}{c}\text{by Lemmas 5.1} \\ \text{and 5.2}\end{array}\right)$$

The result now follows by repeating the argument for $T_3, T_4, \dots$

We now introduce $S_n := \sum_{i=1}^{n} T_i$, $n = 1, 2, \dots$. By Proposition 5.4, S_n is a sum of independent identically distributed exponential random variables,

$$S_n \sim \text{Gamma}(n, \lambda), \quad n = 1, 2, \dots.$$

Theorem 5.1 Assume that $\{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda > 0$. Then, for each $t \geq 0$, $N(t)$ is a Poisson random variable with rate λt. That is:

$$\mathbb{P}(N(t)=n) = \frac{(\lambda t)^n}{n!} e^{-\lambda}, \qquad n=0,1,2,\dots$$

Proof By Lemma 5.2, we have:

$$\mathbb{P}(N(t)=0) = e^{-\lambda t} = \frac{(\lambda t)^0}{0!} e^{-\lambda t}.$$

In order to determine $\mathbb{P}(N(t)=n)$, we condition on the event $S_n = s$.

If $s>t$, we obviously have that

$$\mathbb{P}(N(t)=n \mid S_n = s) = 0.$$

(probability that $N(t)=n$ given that the Poisson process reaches n for the first time at time s)

Consider $0<s\leq t$:

$$\mathbb{P}(N(t)=n \mid S_n=s) = \mathbb{P}(T_{n+1} > t-s \mid S_n = s) = \left(\begin{matrix} T_{n+1} \text{ and } S_n = \sum_{i=1}^{n} T_i \\ \text{are independent} \end{matrix}\right) =$$

$$= \mathbb{P}(T_{n+1} > t-s) = e^{-\lambda(t-s)},$$

hence

$$\mathbb{P}(N(t)=n) = \left(\begin{matrix} \text{continuous version of the law} \\ \text{of total probability} \end{matrix}\right) =$$

$$= \int_0^{\infty} \mathbb{P}(N(t)=n \mid S_n=s)\cdot f_{S_n}(s)\, ds$$

$$= \int_0^{t} \mathbb{P}(N(t)=n \mid S_n=s)\cdot f_{S_n}(s)\, ds$$

$$= \int_0^{t} e^{-\lambda(t-s)} \frac{\lambda^n}{\Gamma(n)} s^{n-1} e^{-\lambda s}\, ds =$$

$$= \frac{\lambda^n}{(n-1)!} e^{-\lambda t} \cdot \int_0^{t} s^{n-1}\, ds =$$

$$= \frac{(\lambda t)}{n!} e^{-\lambda t}, \qquad n=1,2,\dots$$

Example 5.13 Suppose that people immigrate into a territory according to a Poisson process with rate $\lambda = 2$ per day.

(a) Find the probability that there are 10 arrivals in the following 7 days.

(b) Find the expected number of days until there have been 20 arrivals.

Solution

(a) We recall that $\mathbb{P}(N(t)=n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$, hence

$$\mathbb{P}(N(7)=10) = \frac{(2\cdot 7)^{10}}{10!} e^{-2\cdot 7} \approx 0.06628.$$

(b) We recall that

$S_n = \sum_{i=1}^{n} T_i$ = The point of time of the n-th arrival $\sim$ Gamma(n,λ).

Hence we get $\mathbb{E}[S_n] = \sum_{i=1}^{n} \mathbb{E}[T_i] = \frac{n}{\lambda}$.

In particular, $\mathbb{E}[S_{20}] = \frac{20}{2} = 10$

3. Further properties of the Poisson process

Consider a Poisson process $\{N(t), t \geq 0\}$ with rate λ. Assume that N "counts" some events, and each event can be classified as either a Type I event or Type 2 event.

$$p := \mathbb{P}(\text{A given event is of Type I}),$$

$$1-p := \mathbb{P}(\text{A given event is of Type II}).$$

We then introduce:

$N_1(t)$ = The number of type I events in $(0,t]$,

$N_2(t)$ = The number of type II events in $(0,t]$.

Remark $N(t) = N_1(t) + N_2(t)$ and, for $s<t$ we have:

$$N_1(t) - N_1(s) \mid N(t) - N(s) \sim \text{Bin}(n,p)$$

$$N_2(t) - N_2(s) \mid N(t) - N(s) \sim \text{Bin}(n, 1-p)$$

From this we get:

$$\mathbb{P}(N_1(t) - N_1(s) = k) = \sum_{n=k}^{\infty} \mathbb{P}(N_1(t) - N_1(s) = k \mid N(t) - N(s) = n) \cdot \mathbb{P}(N(t) - N(s) = n) =$$

$$= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \frac{(\lambda(t-s))^n}{n!} e^{-\lambda(t-s)} =$$

$$= \sum_{n=k}^{\infty} \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \frac{(\lambda(t-s))^k}{n!} (\lambda(t-s))^{n-k} e^{-\lambda p(t-s)} e^{-\lambda(1-p)(t-s)} =$$

$$= \frac{(\lambda p(t-s))^k}{k!} e^{-\lambda p(t-s)} \sum_{n=k}^{\infty} \frac{(\lambda(1-p)(t-s))^{n-k}}{(n-k)!} e^{-\lambda(1-p)(t-s)} =$$

$$= \frac{(\lambda p(t-s))^k}{k!},$$

i.e. $N_1(t) - N_1(s) \sim \mathrm{Poiss}(\lambda p(t-s))$. Similarly, $N_2(t) - N_2(s) \sim \mathrm{Poiss}(\lambda(1-p)(t-s))$.

Proposition $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ are both Poisson processes with parameters λp and $\lambda(1-p)$ respectively and both processes are independent.

Proof Since $N(0) = 0$, we have that $N_i(0) = 0$, $i = 1, 2$.

Let $(s_1, t_1]$, $(s_2, t_2]$ be disjoint. Since $\{N(t), t \geq 0\}$ has independent increments, we have for $i = 1, 2$:

$$\mathbb{P}(N_i(t_2) - N_i(s_2) = k \mid N_i(t_1) - N_i(s_1) = \ell)$$

$$= \sum_{n=k}^{\infty} \mathbb{P}(N_i(t_2) - N_i(s_2) = k \mid N(t_2) - N(s_2) = n, N_i(t_1) - N_i(s_1) = \ell) \times$$

$$\times \mathbb{P}(N(t_2) - N(s_2) = n \mid N_i(t_1) - N_i(s_1) = \ell)$$

$$= \sum_{n=k}^{\infty} \mathbb{P}(N_i(t_2) - N_i(s_2) = k \mid N(t_2) - N(s_2) = n)\, \mathbb{P}(N(t_2) - N(s_2) = n)$$

$$= \mathbb{P}(N_i(t_2) - N_i(s_2) = k),$$

therefore $\{N_i(t), t \geq 0\}$ has independent increments.

Next, by the Remark above,

$$\mathbb{P}(N_1(t+h) - N_1(t) = 1) = \frac{(\lambda p h)^1}{1!} e^{-\lambda p h} = \lambda p h\, e^{-\lambda p h} = \lambda p h + o(h)$$

since $\frac{\lambda p h\, e^{-\lambda p h} - \lambda p h}{h} = \lambda p (e^{-\lambda p h} - 1) \to 0$, $h \to 0$, and

$$\mathbb{P}(N_1(t+h) - N_1(t) \geq 2) = 1 - \mathbb{P}(N_1(t+h) - N_1(t) = 1) - \mathbb{P}(N_1(t+h) - N_1(t) = 0)$$

$$= 1 - (\lambda p h + o(h) + e^{-\lambda p h}) = o(h),\ h \to 0$$

since $\frac{1 - \lambda p h - e^{-\lambda p h}}{h} = \left(e^{-\lambda p h} = 1 - \lambda p h + o(h) \text{ by Taylor's formula} \right) =$

$$= \frac{o(h)}{h} \to 0, \quad h \to 0.$$

Similarly, $\mathbb{P}(N_2(t+h) - N_2(t) = 1) = \lambda(1-p)h + o(h)$, $h \to 0$,

$$\mathbb{P}(N_2(t+h)-N_2(t)\geq 2) = o(h), \ h\to 0.$$

Thus $\{N_1(t), t\geq 0\}$ and $\{N_2(t), t\geq 0\}$ are both Poisson processes.

To show that $\{N_1(t), t\geq 0\}$ and $\{N_2(t), t\geq 0\}$ are independent, we let $s<t$ and consider:

$$\begin{aligned}
&\mathbb{P}(\{N_1(t)-N_1(s)=k\}\cap\{N_2(t)-N_2(s)=\ell\}) = \\
&= \mathbb{P}(\{N_1(t)-N_1(s)=k\}\cap\{N(t)-N(s)=k+\ell\}) = \\
&= \mathbb{P}(N_1(t)-N_1(s)=k \mid N(t)-N(s)=k+\ell)\,\mathbb{P}(N(t)-N(s)=k+\ell) \\
&= \binom{k+\ell}{k} p^k (1-p)^\ell \frac{(\lambda(t-s))^{k+\ell}}{(k+\ell)!} e^{-\lambda p(t-s)} \\
&= \frac{(\lambda p(t-s))^k}{k!} e^{-\lambda p(t-s)} \cdot \frac{(\lambda(1-p)(t-s))^\ell}{\ell!} e^{-\lambda(1-p)(t-s)} \\
&= \mathbb{P}(N_1(t)-N_1(s)=k)\,\mathbb{P}(N_2(t)-N_2(s)=\ell),
\end{aligned}$$

i.e. we conclude that $N_1(t)-N_1(s)$ and $N_2(t)-N_2(s)$ are independent random variables for all $s<t$ implying that $\{N_1(t), t\geq 0\}$ and $\{N_2(t), t\geq 0\}$ are independent ◢

4. Alternative definition of the Poisson process

Definition The counting process $\{N(t), t\geq 0\}$ is said to be a (standard) Poisson process with rate $\lambda>0$, if

(i) $N(0)=0$;

(ii) $\{N(t), t\geq 0\}$ has independent increments;

(iii) $\mathbb{P}(N(t+h)-N(t)=1) = \lambda h + o(h), \ h\to 0$;

(iv) $\mathbb{P}(N(t+h)-N(t)\geq 2) = o(h), \ h\to 0$.

It turns out that the definition above is equivalent to the following one:

Definition The counting process $\{N(t), t \geq 0\}$ is said to be a (standard) Poisson process with rate $\lambda > 0$, if

(i') $N(0) = 0$;

(ii') $\{N(t), t \geq 0\}$ has independent increments;

(iii') for any $s < t$ $N(t) - N(s) \sim \mathrm{Poiss}(\lambda(t-s))$, i.e. is a Poisson distributed random variable with parameter $\lambda(t-s)$.

Proposition Two definitions are equivalent.

Proof (i) + (ii) + (iii) + (iv) imply (i') + (ii') + (iii') — see Lemma 5.1 and Theorem 5.1.

(i') + (ii') + (iii') also imply (i) + (ii) + (iii) + (iv) — (i) and (ii) are the same as (i') and (ii') respectively and (iii) gives:

$$\mathbb{P}(N(t+h) - N(t) = 1) = \frac{(\lambda h)^1}{1!} e^{-\lambda h} = \lambda h e^{-\lambda h} = \lambda h + o(h)$$

since $\dfrac{\lambda h e^{-\lambda h} - \lambda h}{h} = \lambda(e^{-\lambda h} - 1) \to 0$, $h \to 0$, and

$$\mathbb{P}(N(t+h) - N(t) \geq 2) = 1 - \mathbb{P}(N(t+h) - N(t) = 1) - \mathbb{P}(N(t+h) - N(t) = 0)$$

$$= 1 - (\lambda h + o(h) + e^{-\lambda h}) = o(h),\ h \to 0$$

since $\dfrac{1 - \lambda h - e^{-\lambda h}}{h} = \left(e^{-\lambda h} = 1 - \lambda h + o(h) \text{ by Taylor's formula} \right) = \dfrac{o(h)}{h} \to 0,$

$h \to 0.$