Definition Let $\{X(t), t \geq 0\}$ be a continuous-time stochastic process with discrete state space $\mathcal{X}$. The process is a Markov chain (Markov process) if for $t>0$:

$$\mathbb{P}(X(s+t)=j \mid X(t)=i, X(t_n)=x_n, X(t_{n-1})=x_{n-1}, \dots, X(t_1)=x_1) =$$
$$= \mathbb{P}(X(s+t)=j \mid X(s)=i),$$

for all $0 \leq t_1 < t_2 < \dots < t_n < s < s+t$ and $i, j, x_1, \dots, x_n \in \mathcal{X}$ s.t. $\mathbb{P}(X(s)=i, X(t_n)=x_n, X(t_{n-1})=x_{n-1}, \dots, X(t_1)=x_1) > 0$.

If we also have that probabilities $\mathbb{P}(X(s+t)=j \mid X(s)=i)$ are independent of s, then such Markov chain is called time-homogeneous (homogeneous, has stationary / homogeneous transition probabilities).

Example Let $\{N(t), t \geq 0\}$ be a Poisson process (homogeneous) with rate λ. This process has independent and stationary increments. Let us show that $\{N(t), t \geq 0\}$ is a Markov chain. Take arbitrary $0 \leq t_1 < t_2 < \dots < t_k < s < t+s$, states $n_1 \leq \dots \leq n_k \leq i \leq j$ (so that $\mathbb{P}(N(t_1)=n_1, \dots, N(t_k)=n_k, N(s)=i) > 0$) and consider

$$\mathbb{P}(N(s+t)=j \mid N(t_1)=n_1, \dots, N(t_k)=n_k, N(s)=i) =$$
$$= \frac{\mathbb{P}(N(s+t)=j, N(t_1)=n_1, \dots, N(t_k)=n_k, N(s)=i)}{\mathbb{P}(N(t_1)=n_1, \dots, N(t_k)=n_k, N(s)=i)} =$$
$$= \frac{\mathbb{P}(N(t_1)=n_1, N(t_2)-N(t_1)=n_2-n_1, \dots, N(t_k)-N(t_{k-1})=n_k-n_{k-1}, N(s)-N(t_k)=i-n_k, N(s+t)-N(s)=j-i)}{\mathbb{P}(N(t_1)=n_1, N(t_2)-N(t_1)=n_2-n_1, \dots, N(t_k)-N(t_{k-1})=n_k-n_{k-1}, N(s)-N(t_k)=i-n_k)}$$

$$= \frac{\mathbb{P}(N(t_1)=n_1)\cdot \ldots \cdot \mathbb{P}(N(s)-N(t_k)=i-n_k)\ \mathbb{P}(N(s+t)-N(s)=j-i)}{\mathbb{P}(N(t_1)=n_1)\cdot \ldots \cdot \mathbb{P}(N(s)-N(t_k)=i-n_k)} =$$

$$= \mathbb{P}(N(s+t)-N(s)=\underbrace{j-i}_{\geq 0}) = \frac{(\lambda t)^{j-i}}{(j-i)!} e^{-\lambda t}$$ — does not depend on s.

Similarly,

$$\mathbb{P}(N(t+s)=j \mid N(s)=i) = \frac{(\lambda t)^{j-i}}{(j-i)!} e^{-\lambda t}.$$

This implies that $\{N(t), t\geq 0\}$ is a continuous-time Markov chain ◢

Remark In the example above we found

$$\mathbb{P}(N(t+s)=j \mid N(s)=i)$$

for $j \geq i$. Obviously, since $N(t)$, $t\geq 0$, is a non-decreasing process (by definition, it is a counting process)

$$\mathbb{P}(N(t+s)=j \mid N(s)=i) = 0,$$

if $i > j$. Whence we have that for all $t,s \geq 0$

$$\mathbb{P}(N(t+s)=j \mid N(s)=i) = \begin{cases} \frac{(\lambda t)^{j-i}}{(j-i)!} e^{-\lambda t}, & \text{if } j \geq i, \\ 0, & \text{if } j < i. \end{cases}$$

◢

Consider $\{X(t), t\geq 0\}$ - homogeneous Markov chain and assume that $X(0)=i$,

$$T_i := \inf\{u\geq 0 : X(u)\neq i\},$$

i.e. T_i is the point of time when the process leaves state i. We then set $s,t>0$ and consider

$$\begin{aligned}
\mathbb{P}(T_i > s+t \mid T_i > s) &= \mathbb{P}(X(u)=i,\ 0\leq u\leq s+t \mid X(u)=i,\ 0\leq u\leq s) \\
&= \mathbb{P}(X(u)=i,\ s\leq u\leq s+t \mid X(u)=i,\ 0\leq u\leq s) \\
&= \mathbb{P}(X(u)=i,\ s\leq u\leq s+t \mid X(s)=i) \quad (\text{Markov property}) \\
&= \mathbb{P}(X(u)=i,\ 0\leq u\leq t \mid X(0)=i) \quad (\text{homogeneous}) \\
&= \mathbb{P}(X(u)=i,\ 0\leq u\leq t) \\
&= \mathbb{P}(T_i > t).
\end{aligned}$$

This implies that T_i is memoryless $\Rightarrow$ [Lecture 14] $\Rightarrow$

$\Rightarrow$ T_i is exponentially distributed.

Assume that $X(r)=i$ and define

$$T_i := \inf\{u\geq 0 : X(r+u)\neq i\},$$

i.e T_i+r is the moment of time when $\{X(t), t\geq 0\}$ leaves the state i. Then, just as above,

$\mathbb{P}(T_i > s+t \mid T_i>s) = \mathbb{P}(T_i>t) \Rightarrow T_i$ is also an exponential r.v.

Definition (alternative definition of a Markov chain)

A continuous-time Markov chain with stationary (homogeneous) transition probabilities and discrete state space $\mathfrak{X}$ is a stochastic process such that:

(i) the times spent in different states are independent random variables (because of Markov property);

(ii) the amount of time spent in state $i \in \mathcal{X}$ is exponentially distributed with some mean ν_i^{-1} (because of Markov property and stationarity);

(iii) when the process leaves state i, it enters state j with some transition probability $Q_{i,j}$, where

$$Q_{i,i} = 0, \quad i \in \mathcal{X},$$
$$\sum_{j \in \mathcal{X}} Q_{i,j} = 1, \quad i \in \mathcal{X}.$$

(iv) Transitions follow a discrete-time Markov chain.

Example 6.1 (A shoe Shine Shop) Consider a Markov chain with state space $\mathcal{X} = \{0, 1, 2\}$, where

- State 0: No customer;
- State 1: Customer sits on chair 1, where their shoes are cleaned and the polish is applied;
- State 2: Customer sits on chair 2, where the polish is buffed.

- $X(s) = 0$: in this state, customers arrive in accordance to a Poisson process with rate λ. The time spent in this state is $T_0 \sim \exp(\lambda)$. Then the process transits to state 1 with probability $Q_{0,1} = 1$.

- $X(t)=1$: The time spent in this state is $T_1 \sim \exp(\mu_1)$. Then the process transits to state 2 with probability $Q_{1,2}=1$.

- $X(u)=1$: The time spent in this state is $T_2 \sim \exp(\mu_2)$. Then the process transits to state 0 with probability $Q_{2,0}=1$ and the process repeats the same cycle.

$$Q = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

the built-in discrete time Markov chain is periodic with a period 3. Despite that, the continuous time Markov chain $\{X(t), t\geq 0\}$ will have a well-defined limiting distribution.

6.3. Birth and Death processes

A Birth and Death Process $\{X(t), t \geq 0\}$ has state space $\mathcal{X} = \{0, 1, 2, \dots\}$. Assume that $X(t) = n > 0$. Then the next transition is determined as follows:

- Sample random variables $V \sim \exp(\lambda_n)$, $W \sim \exp(\mu_n)$ independent of each other with respective outcomes v and w;
- if $v < w$, then the process transits to state $n+1$ at time $t+v$, i.e. $X(t+v) = n+1$ (this is called a birth);
- if $w < v$, then the process transits to state $n-1$ at time $t+w$, i.e. $X(t+w) = n-1$ (this is called a death).

When $X(t) = 0$, only births are possible, so in this case we assume that $W = \infty$, which corresponds to the rate μ_0 being zero and $P_{0,1} = 1$.

The transition (either a birth or a death) happens at time $U = \min(V, W) \sim \exp(\lambda_n + \mu_n)$. Moreover the transition probabilities for the built-in discrete-time Markov chain

$$P_{n,n+1} = \mathbb{P}(V < W) = \frac{\lambda_n}{\lambda_n + \mu_n}, \quad P_{n,n-1} = \mathbb{P}(W < V) = \frac{\mu_n}{\lambda_n + \mu_n}.$$

Example 6.2 (Pure birth process) Assume that $\{X(t), t \geq 0\}$ is a birth and death process with

$$\mu_n = 0, \quad n \geq 0.$$
$$\lambda_n = \lambda, \quad n \geq 0.$$

Since the death rate is zero, this is a pure birth process with constant birth rate λ. This implies that the time between transitions is exponentially distributed with rate $\lambda > 0 \implies \{X(t), t \geq 0\}$ is a Poisson process with rate λ.

Example 6.4 Assume that $\{X(t), t \geq 0\}$ is a birth and death process with

$$\mu_n = \mu n, \quad \text{for all } n \geq 1.$$
$$\lambda_n = \lambda n + \theta, \quad \text{for all } n \geq 1.$$

That can be interpreted as follows: we have some population of size $X(t)$ at moment t; each member dies with rate μ; produces offspring with rate λ and there is also migration with rate θ.

We assume that $X(0) = i$ and introduce

$$M(t) = \mathbb{E}[X(t)].$$

We want to find $M(t)$ by solving a differential equation. Observe that

$$M(t+h) = \mathbb{E}[X(t+h)] = \mathbb{E}[\,\mathbb{E}[X(t+h) \mid X(t)]\,].$$

Since the time between transitions is exponentially distributed, the probability of more then one transition in an interval of length h is $o(h)$. Hence we have:

$$\mathbb{P}(X(t+h) = X(t)+1 \mid X(t)) = [X(t)\lambda + \theta]\cdot h + o(h)$$

$$\mathbb{P}(X(t+h) = X(t)-1 \mid X(t)) = X(t)\mu h + o(h),$$

$$\mathbb{P}(X(t+h) = X(t) \mid X(t)) = 1 - (X(t)\lambda + \theta + X(t)\mu)h + o(h).$$

From this, it follows that

$$\begin{aligned}\mathbb{E}[X(t+h) \mid X(t)] &= (X(t)+1)(X(t)\lambda+\theta)\cdot h + (X(t)-1)X(t)\mu h + \\ &\quad + X(t) - X(t)(X(t)\lambda + \theta + X(t)\mu)h + o(h) \\ &= X(t)\lambda h + \theta h - X(t)\mu h + X(t) + o(h) \\ &= X(t) + (\lambda-\mu)X(t)h + \theta h + o(h),\end{aligned}$$

i.e.

$$M(t+h) - M(t) = (\lambda-\mu)M(t)h + \theta h + o(h)$$

$$M'(t) = (\lambda-\mu)M(t) + \theta.$$

This is a standard linear ODE that can be easily solved: if $\lambda \neq \mu$, then

$$M(t) = -\frac{\theta}{\lambda-\mu} + Ce^{(\lambda-\mu)t},$$ where C is some constant. In order to compute it, we note that $M(0) = i \Rightarrow C = i + \frac{\theta}{\lambda-\mu}$, i.e.

$$M(t) = -\frac{\theta}{\lambda-\mu} + \left(i + \frac{\theta}{\lambda-\mu}\right)e^{(\lambda-\mu)t} = \frac{\theta}{\lambda-\mu}\left(e^{(\lambda-\mu)t} - 1\right) + ie^{(\lambda-\mu)t}.$$

If $\lambda = \mu$, the differential equation becomes

$$M'(t) = \theta \implies M(t) = \theta t + i$$