UiO : Department of Mathematics
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Lecture 10. Gambler's ruin problem. Mean time in transient states

STK2130 – Modellering av stokastiske prosesser

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Brush-up

Definition

If a Markov chain can only return to a state in a multiple of $d > 1$ steps, it is said to be *periodic*. A Markov chain which is not periodic is said to be *aperiodic*. An irreducible, positive recurrent, aperiodic Markov chain is said to be *ergodic*.

Theorem

If a Markov chain with state space $\mathcal{S}$ is ergodic (i.e. irreducible + positive recurrent + aperiodic), then the limiting probabilities will always exist, and do not depend on the initial state, and we have:

$$\lim_{n\to\infty} P_{i,j}^n = \pi_j, \quad \textit{for all } i, j \in \mathcal{S}.$$

Brush-up and Remark

We have that for ergodic Markov chains the *transition* (i.e. conditional) probabilities behave as follows:

$$\lim_{n\to\infty} P^n_{i,j} = \pi_j, \quad \text{for all } i, j \in \mathcal{S}.$$

But we also know the following about the *unconditional* probabilities $\pi_j^{(n)} := \mathbb{P}(X_n = j)$: **IF** the limit $\lim_{n\to\infty} \pi_j^{(n)}$ **AND IF** the conditions of Theorem 4.1 are satisfied (i.e. long-run proportions π_j exist), then

$$\lim_{n\to\infty} \pi_j^{(n)} = \pi_j, \quad j \in \mathcal{S}.$$

However, when we stated this, we **DID NOT KNOW** when the limit $\lim_{n\to\infty} \pi_j^{(n)}$ exists.

Exercise

Prove that, for ergodic Markov chain with any initial distribution, the limits of the form $\lim_{n\to\infty} \pi_j^{(n)}$ exist and

$$\lim_{n\to\infty} \pi_j^{(n)} = \pi_j, \quad j \in \mathcal{S}.$$

The Gambler's Ruin Problem

Vanilla gambler's ruin problem

Consider a gambler who at each play of the game has probability p of winning one unit and probability $q = 1 - p$ of losing one unit. Assuming that successive plays of the game are independent, what is the probability that, starting with i units, the gambler's fortune will reach N before reaching 0?

The Gambler's Ruin Problem

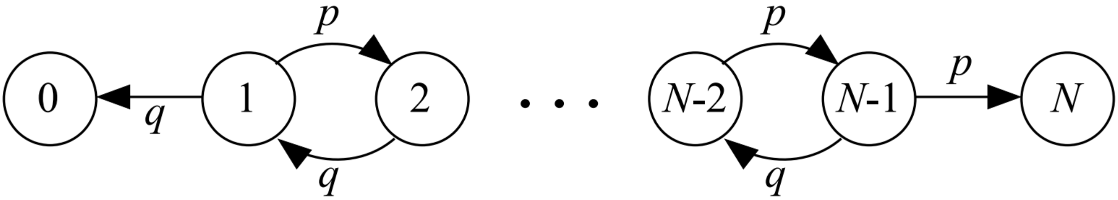
What we have:

- potentially infinite sequence of independent identically distributed games;
- $\mathbb{P}(\text{Win one unit}) = p$, $\mathbb{P}(\text{Lose one unit}) = q = 1 - p$;
- $X_n =$ The player's fortune after n games, $n = 0, 1, 2, \ldots$;
- state space: $\mathcal{S} = \{0, 1, \ldots, N\}$ representing the player's fortune;
- transition probabilities:

$$\begin{aligned} P_{0,0} = P_{N,N} = 1, \\ P_{i,i+1} = p, \quad i = 1, 2, \ldots, N-1, \\ P_{i,i-1} = q, \quad i = 1, 2, \ldots, N-1; \end{aligned}$$

- Classes: $\{0\}$ (recurrent), $\{1, 2, \ldots, N-1\}$ (transient), $\{N\}$ (recurrent).

The Gambler's Ruin Problem



We then introduce:

$$P_i := \mathbb{P}\left(\bigcup_{n=0}^{\infty}\{X_n = N\} \;\middle|\; X_0 = i\right), \quad i = 0, 1, 2, ..., N,$$

– the probability that, starting with i, the gambler's fortune will eventually reach N.

Remark

Before, we have seen probabilities of such type – we denoted them as $f_{i,j}$.

The Gambler's Ruin Problem

By conditioning on X_1, we obtain:

$$P_i = pP_{i+1} + qP_{i-1}, \quad i = 1, 2, ..., N-1.$$

Since $p + q = 1$, we may alternatively write:

$$pP_i + qP_i = pP_{i+1} + qP_{i-1}, \quad i = 1, 2, ..., N-1$$

or

$$qP_i - qP_{i-1} = pP_{i+1} - pP_i, \quad i = 1, 2, ..., N-1.$$

From this we get

$$P_{i+1} - P_i = \frac{q}{p}(P_i - P_{i-1}), \quad i = 1, 2, ..., N-1.$$

The Gambler's Ruin Problem

Since $P_0 = 0$ we get the following:

$$P_2 - P_1 = \frac{q}{p}(P_1 - P_0) = \frac{q}{p}P_1$$

$$P_3 - P_2 = \frac{q}{p}(P_2 - P_1) = \left(\frac{q}{p}\right)^2 P_1$$

$$\vdots$$

$$P_i - P_{i-1} = \frac{q}{p}(P_{i-1} - P_{i-2}) = \left(\frac{q}{p}\right)^{i-1} P_1$$

$$\vdots$$

$$P_N - P_{N-1} = \frac{q}{p}(P_{N-1} - P_{N-2}) = \left(\frac{q}{p}\right)^{N-1} P_1.$$

The Gambler's Ruin Problem

We then add the first $(i-1)$ equations:

$$
\begin{aligned}
(P_2 - P_1) + (P_3 - P_2) + \ + (P_i - P_{i-1}) \\
= P_i - P_1 = \left(\left(\frac{q}{p}\right) + \left(\frac{q}{p}\right)^2 + ... + \left(\frac{q}{p}\right)^{i-1}\right) P_1
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
P_i &= \left(1 + \left(\frac{q}{p}\right) + \left(\frac{q}{p}\right)^2 + ... + \left(\frac{q}{p}\right)^{i-1}\right) P_1 \\
&= \begin{cases} \frac{1-(q/p)^i}{1-(q/p)} P_1, & \text{if } \frac{q}{p} \neq 1, \\ iP_1, & \text{if } \frac{q}{p} = 1. \end{cases}
\end{aligned}
$$

The Gambler's Ruin Problem

Now, we use that $(q/p) \neq 1$ if and only if $p \neq \frac{1}{2}$, and that $P_N = 1$.

- **Case 1**: $p \neq \frac{1}{2}$.

$$P_N = 1 = \frac{1-(q/p)^N}{1-(q/p)} P_1$$

  Hence, in this case

$$P_1 = \frac{1-(q/p)}{1-(q/p)^N}.$$

- **Case 2**: $p = \frac{1}{2}$.

$$P_N = 1 = NP_1.$$

  Hence, in this case:

$$P_1 = \frac{1}{N}.$$

The Gambler's Ruin Problem

By inserting the expression for P_1 into the formula P_i we get:

$$P_i = \begin{cases} \frac{1-(q/p)^i}{1-(q/p)} P_1, & \text{if } \frac{q}{p} \neq 1, \\ iP_1, & \text{if } \frac{q}{p} = 1, \end{cases}$$
$$= \begin{cases} \frac{1-(q/p)^i}{1-(q/p)^N}, & \text{if } p \neq \frac{1}{2}, \\ \frac{i}{N}, & \text{if } p = \frac{1}{2}. \end{cases}$$

Note that if $p > \frac{1}{2}$ then $(q/p) < 1$, and hence $(q/p)^N \to 0$ as $N \to 0$. Similarly, if $p < \frac{1}{2}$ then $(q/p) > 1$, and hence $(q/p)^N \to \infty$. Thus:

$$\lim_{N \to \infty} P_i = \begin{cases} 1 - (q/p)^i, & \text{if } p > \frac{1}{2}, \\ 0, & \text{if } p \leq \frac{1}{2}. \end{cases}$$

Example 4.30. Penny flipping

Problem

Suppose Max and Patty decide to flip pennies; the one coming closest to the wall wins. Patty, being the better player, has a probability 0.6 of winning on each flip.

(a) If Patty starts with five pennies and Max with ten, what is the probability that Patty will wipe Max out?

(b) What if Patty starts with 10 and Max with 20?

Example 4.30. Penny flipping

We assume that $p = \mathbb{P}(\text{Patty wins}) = 0.6$ and that $q = \mathbb{P}(\text{Max wins}) = 0.4$. Hence, $(q/p) = 0.4/0.6 = \frac{2}{3}$. Moreover, we let X_n be the number of pennies owned by Patty after n plays.

(a) $X_0 = 5$, $N = 5 + 10 = 15$.

$$P_5 = \frac{1 - \left(\frac{2}{3}\right)^5}{1 - \left(\frac{2}{3}\right)^{15}} \approx 0.87.$$

(b) $X_0 = 10$, $N = 10 + 20 = 30$.

$$P_{10} = \frac{1 - \left(\frac{2}{3}\right)^{10}}{1 - \left(\frac{2}{3}\right)^{15}} \approx 0.98.$$

"Drug racing"

"Drug racing"

Suppose that two new drugs have been developed for treating a certain disease. Drug i has a cure rate α_i, $i = 1, 2$, in the sense that each patient treated with drug i will be cured with probability α_i . These cure rates are not known.

Experiment

Pairs of patients are treated sequentially with one member of the pair receiving drug 1 and the other drug 2. The results for each pair are determined. The testing stops when the cumulative number of cures using one of the drugs exceeds the cumulative number of cures when using the other by some fixed predetermined number, M.

NB!

Only pairs where the result for the patient who receives drug 1 is different from the result for the patient who receives drug 2 are included in the analysis.

“Drug racing”

$$\alpha_i = \mathbb{P}(\text{A patient receiving drug number } i \text{ is cured}), \quad i = 1, 2.$$

Consider the *n*-th pair where the result is different for the two drugs. Then:

$$\begin{aligned} p &= P\left((\text{Drug 1 works}) \cap (\text{Drug 2 fails}) \mid \text{Different result}\right) \\ &= \frac{\alpha_1(1-\alpha_2)}{\alpha_1(1-\alpha_2) + (1-\alpha_1)\alpha_2} \end{aligned}$$

$$\begin{aligned} q &= \mathbb{P}\left((\text{Drug 1 fails}) \cap (\text{Drug 2 works}) \mid \text{Different result}\right) \\ &= \frac{(1-\alpha_1)\alpha_2}{\alpha_1(1-\alpha_2) + (1-\alpha_1)\alpha_2}. \end{aligned}$$

“Drug racing”

We then introduce

$$X_n = \text{The number of cured patients receiving drug 1 among the first n pairs} \\ - \text{The number of cured patients receiving drug 2 among the first n pairs}$$

Then $\{X_n\}$ is a Markov chain with state space

$$\mathcal{S} = \{-M, -(M-1), ..., -1, 0, 1, ..., (M-1), M\}$$

and transition probabilities:

$$\begin{aligned} P_{-M,-M} = P_{M,M} = 1, \\ P_{i,i+1} = p, \quad i = -(M-1), ..., (M-1), \\ P_{i,i-1} = q, \quad i = -(M-1), ..., (M-1) \end{aligned}$$

- Classes: $\{-M\}$ (recurrent), $\{-(M-1), ..., (M-1)\}$ (transient), $\{M\}$ (recurrent).
- If the chain is absorbed in state M, drug 1 wins the race.
- If the chain is absorbed in state $-M$, drug 2 wins the race.

“Drug racing”

Alternatively, let $Y_n = X_n + M$. Then $\{Y_n\}$ is a Markov chain with:

$$\mathcal{S} = \{0, 1, ..., (M-1), M, (M+1), ..., (2M-1), 2M\}$$

and transition probabilities:

$$\begin{aligned} P_{0,0} = P_{2M,2M} &= 1, \\ P_{i,i+1} = p, &\quad i = 1, ..., (2M-1), \\ P_{i,i-1} = q, &\quad i = 1, ..., (2M-1) \end{aligned}$$

- Classes: $\{0\}$ (recurrent), $\{1, ..., (2M-1)\}$ (transient), $\{2M\}$ (recurrent).
- If the chain is absorbed in state $2M$, drug 1 wins the race.
- If the chain is absorbed in state 0, drug 2 wins the race.

“Drug racing”

Warning!

Of course, if, say, drug 1 wins, we cannot really claim that $\alpha_1 > \alpha_2$: it still may happen that $\alpha_1 < \alpha_2$ and drug 1 won “by accident”. Let us analyze the corresponding probabilities.

Assume that $X_0 = 0$ or equivalently that $Y_0 = X_0 + M = M$. We then have:

$$\mathbb{P}(\text{Test asserts that drug 1 is best} \mid X_0 = 0) = \mathbb{P}(\text{Test asserts that drug 1 is best} \mid Y_0 = M)$$
$$= \frac{1-(q/p)^M}{1-(q/p)^{2M}} = \frac{1-(q/p)^M}{(1-(q/p)^M)(1+(q/p)^M)} = \frac{1}{1+(q/p)^M}.$$

Similarly,

$$\mathbb{P}(\text{Test asserts that drug 2 is best} \mid X_0 = 0) = \frac{1}{1+(p/q)^M}.$$

- If $p > q$ (this happens when $\alpha_1 > \alpha_2$), then the probability that drug 1 wins $\to 1$ as $M \to \infty$.
- If $p < q$ (this happens when $\alpha_1 < \alpha_2$), then the probability that drug 1 wins $\to 0$ as $M \to \infty$.
- If $p = q = \frac{1}{2}$ (this can happen only when $\alpha_1 = \alpha_2$) then both probabilities above are $\frac{1}{2}$.

“Drug racing”

Assume that $\alpha_1 = 0.6$, $\alpha_2 = 0.4$ and $M = 5$. Thus, drug 1 is the best drug. Then we have:

$$\alpha_1(1-\alpha_2) = 0.6^2 = 0.36, \qquad \alpha_2(1-\alpha_1) = 0.4^2 = 0.16.$$

Hence, we have:

$$p = \frac{\alpha_1(1-\alpha_2)}{\alpha_1(1-\alpha_2)+(1-\alpha_1)\alpha_2} = \frac{0.36}{0.36+0.16} = 0.6923,$$
$$q = \frac{\alpha_2(1-\alpha_1)}{\alpha_1(1-\alpha_2)+(1-\alpha_1)\alpha_2} = \frac{0.16}{0.36+0.16} = 0.3077.$$

From this we get that

$$\mathbb{P}(\text{Test asserts that drug 1 is best} \mid X_0 = 0) = \frac{1}{1+(q/p)^5} = \frac{1}{1+(0.3077/0.6923)^5} = 0.9830,$$
$$\mathbb{P}(\text{Test asserts that drug 2 is best} \mid X_0 = 0) = \frac{1}{1+(p/q)^5} = \frac{1}{1+(0.6923/0.3077)^5} = 0.0170.$$

"Drug racing"

If we increase M to 10, we get that

$$\mathbb{P}(\text{Test asserts that drug 1 is best} \mid X_0 = 0) = \frac{1}{1 + (q/p)^{10}} = \frac{1}{1 + (0.3077/0.6923)^{10}} = 0.9997,$$
$$\mathbb{P}(\text{Test asserts that drug 2 is best} \mid X_0 = 0) = \frac{1}{1 + (p/q)^{10}} = \frac{1}{1 + (0.6923/0.3077)^{10}} = 0.0003.$$

Mean time spent in transient states

Proposition (improved version of Corollary 4.2 from Lecture 5)

Let $i \in \mathcal{S}$ be a recurrent state and $i \to j$. Then j is also recurrent.

Proof. It is sufficient to prove that for any $j \in \mathcal{S}$, $j \neq i$,

$$\text{(recurrence of } i) \quad + \quad (i \to j) \quad \Longrightarrow \quad j \to i,$$

because then $i \leftrightarrow j$ and we can use Corollary 4.2. Assume that $X_0 = i$ and that i is **not** accessible from j. It is easy to see that $i \to j$ implies that there exist $n > 0$ and states s_1, ..., $s_{n-1} \in \mathcal{S}$ such that $s_r \neq i$ for all $r = 1, ..., n-1$ and $P_{i,s_1} \cdot P_{s_1,s_2} \cdot ... \cdot P_{s_{n-1},j} > 0$. Then

$$\begin{aligned} 1 - f_i &= 1 - \mathbb{P}\left(\bigcup_{r=1}^{\infty}\{X_r = i\} \,\middle|\, X_0 = i\right) = \mathbb{P}\left(\bigcap_{r=1}^{\infty}\{X_r \neq i\} \,\middle|\, X_0 = i\right) \\ &\geq \mathbb{P}\left(\{X_1 = s_1\} \cap \{X_2 = s_2\} \cap ... \cap \{X_{n-1} = s_{n-1}\} \cap \{X_n = j\} \cap \bigcap_{r=n+1}^{\infty}\{X_r \neq i\} \,\middle|\, X_0 = i\right) \\ &= P_{i,s_1} \cdot P_{s_1,s_2} \cdot ... \cdot P_{s_{n-1},j}\mathbb{P}\left(\bigcap_{r=n+1}^{\infty}\{X_r \neq i\} \,\middle|\, X_n = j\right) > 0. \end{aligned}$$

Thus $1 - f_i > 0$, $f_i < 1 \Longrightarrow i$ is not recurrent. Contradiction.

Mean time spent in transient states

Consider a finite state Markov chain $\{X_n\}$ with state space $\mathcal{S}$, and with transient states $\mathcal{T} = \{1, 2, ..., t\} \subset \mathcal{S}$, and let the transition probabilities between the transient states be:

$$\mathbf{P}_T = \begin{pmatrix} P_{1,1} & P_{1,2} & \cdots & P_{1,t} \\ \vdots & \vdots & \ddots & \vdots \\ P_{t,1} & P_{t,2} & \cdots & P_{t,t} \end{pmatrix}$$

Remark

Since $\mathbf{P}_T$ is only a submatrix of the full matrix of transition probabilities, the row sums in $\mathbf{P}_T$ are less than 1.

We then introduce for all $i, j \in \mathcal{T}$:

$$s_{i,j} = \mathbb{E}\left[\text{Number of periods in state } j \mid X_0 = i\right]$$
$$\delta_{i,j} = \mathbb{1}(i = j)$$

Remark

From what we proved before, $s_{i,j} = 0$, if i is recurrent and j is transient.

Mean time spent in transient states

By conditioning on the initial transition we get for all $i, j \in \mathcal{T}$:

$$s_{i,j} = \delta_{i,j} + \sum_{k \in \mathcal{S}} P_{i,k} s_{k,j} = \delta_{i,j} + \sum_{k \in \mathcal{T}} P_{i,k} s_{k,j}, \tag{1}$$

where we have used that $s_{k,j} = 0$ if $k \in \mathcal{S} \setminus T$.
We then let $\mathbf{I}$ be the identity matrix of size t, and let

$$\mathbf{S} = \begin{pmatrix} s_{1,1} & s_{1,2} & \cdots & s_{1,t} \\ \vdots & \vdots & \ddots & \vdots \\ s_{t,1} & s_{t,2} & \cdots & s_{t,t} \end{pmatrix}$$

Then (1) can be written in matrix notation as:

$$\mathbf{S} = \mathbf{I} + \mathbf{P}_T \mathbf{S}.$$

Mean time spent in transient states

This last equation can be rewritten as

$$\mathbf{S} - \mathbf{P}_T\mathbf{S} = (\mathbf{I} - \mathbf{P}_T)\mathbf{S} = \mathbf{I}.$$

We then multiply both sides of the last equation by $(\mathbf{I} - \mathbf{P}_T)^{-1}$ (it exists) and get:

$$\mathbf{S} = (\mathbf{I} - \mathbf{P}_T)^{-1}.$$

That is, we can find $s_{i,j}$ for all $i, j \in \mathcal{T}$ by inverting the matrix $(\mathbf{I} - \mathbf{P}_T)$.

Example 4.32

We consider the gambler's ruin problem with $p = 0.4$, $q = 0.6$, $N = 7$ and $X_0 = 3$. We want to determine:

- $s_{3,5}$ = The expected number of times the player has 5 units,
- $s_{3,2}$ = The expected number of times the player has 2 units.

In this case we have $\mathcal{T} = \{1, 2, ..., 6\}$. The transition probabilities for this Markov chain is:

$$
\begin{aligned}
P_{i,i} &= 1, \quad i \in \mathcal{S} \setminus \mathcal{T}, \\
P_{i,i} &= 0, \quad i \in \mathcal{T}, \\
P_{i,i+1} &= 0.4, \quad i \in \mathcal{T}, \\
P_{i,i-1} &= 0.6, \quad i \in \mathcal{T}.
\end{aligned}
$$

Example 4.32

$$\mathbf{P}_T = \begin{pmatrix} 0 & 0.4 & 0 & 0 & 0 & 0 \\ 0.6 & 0 & 0.4 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.6 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0.4 \\ 0 & 0 & 0 & 0 & 0.6 & 0 \end{pmatrix}$$

By inverting $(I - \mathbf{P}_T)$, we get:

$$\mathbf{S} = (\mathbf{I} - \mathbf{P}_T)^{-1} = \begin{pmatrix} 1.6149 & 1.0248 & 0.6314 & 0.3691 & 0.1943 & 0.0777 \\ 1.5372 & 2.5619 & 1.5784 & 0.9228 & 0.4857 & 0.1943 \\ 1.4206 & 2.3677 & 2.9990 & 1.7533 & 0.9228 & 0.3691 \\ 1.2458 & 2.0763 & 2.6299 & 2.9990 & 1.5784 & 0.6314 \\ 0.9835 & 1.6391 & 2.0763 & 2.3677 & 2.5619 & 1.0248 \\ 0.5901 & 0.9835 & 1.2458 & 1.4206 & 1.5372 & 1.6149 \end{pmatrix}$$

Hence: $s_{3,5} = 0.9228$ and $s_{3,2} = 2.3677$.

Probability of transitions into transient states

For all $i, j \in \mathcal{T}$ we introduce:

$$f_{i,j} = \mathbb{P}(\text{At least one transition into state } j | X_0 = i)$$

Then we have:

$$\begin{aligned} s_{i,j} &= \mathbb{E}\left[\text{Periods in } j \mid X_0 = i, \text{ At least one trans. into } j\right] f_{i,j} \\ &\quad + \mathbb{E}\left[\text{Periods in } j \mid X_0 = i, \text{ No trans. into } j\right](1 - f_{i,j}) \\ &= (\delta_{i,j} + s_{j,j}) f_{i,j} + \delta_{i,j}(1 - f_{i,j}) \\ &= \delta_{i,j} + f_{i,j} s_{j,j}. \end{aligned}$$

Hence, we find that:

$$f_{i,j} = \frac{s_{i,j} - \delta_{i,j}}{s_{j,j}}, \quad i, j \in \mathcal{T}.$$

Example 4.33

Problem

We consider the gambler's ruin problem with $p = 0.4$, $q = 0.6$, $N = 7$ and $X_0 = 3$. What is the probability that the gambler ever has a fortune of 1?

Solution. We recall that:

$$\mathbf{S} = (\mathbf{I} - \mathbf{P}_T)^{-1} = \begin{pmatrix} 1.6149 & 1.0248 & 0.6314 & 0.3691 & 0.1943 & 0.0777 \\ 1.5372 & 2.5619 & 1.5784 & 0.9228 & 0.4857 & 0.1943 \\ 1.4206 & 2.3677 & 2.9990 & 1.7533 & 0.9228 & 0.3691 \\ 1.2458 & 2.0763 & 2.6299 & 2.9990 & 1.5784 & 0.6314 \\ 0.9835 & 1.6391 & 2.0763 & 2.3677 & 2.5619 & 1.0248 \\ 0.5901 & 0.9835 & 1.2458 & 1.4206 & 1.5372 & 1.6149 \end{pmatrix}$$

and observe that $s_{3,1} = 1.4206$ and $s_{1,1} = 1.6149$. Hence, we get that:

$$f_{3,1} = \frac{s_3}{s_{1,1}} = \frac{1.4206}{1.6149} = 0.8797.$$

Example 4.33

Alternatively, we can compute the probability that the gambler will never have a fortune of 1 (and then subtract it from 1).
In order to implement that approach, consider the Markov chain $\{Y_n\}$ where $Y_n = X_n - 1$, and where we define 0 and 6 as absorbing states for $\{Y_n\}$.
Moreover, we let:

$$P_i = \mathbb{P}\left(\bigcup_{n=0}^{\infty}\{Y_n = 6\} \,\middle|\, Y_n = i\right), \quad i = 1, 2, ..., 6.$$

We recall that:

$$P_i = \frac{1-(q/p)^i}{1-(q/p)^N} = \frac{1-(0.6/0.4)^i}{1-(0.6/0.4)^6}, \quad i = 1, 2, ..., 6.$$

Then it follows that:

$$f_{3,1} = 1 - P_{3-1} = 1 - \frac{1-(0.6/0.4)^2}{1-(0.6/0.4)^6} = 0.8797.$$

Stationary probabilities

We recall that the stationary probabilities $\boldsymbol{\pi}$ of a Markov chain with transition probability matrix $\mathbf{P}$ and state space $\mathcal{S}$ are found as the unique solution of the equations:

$$\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi},$$
$$\sum_{i \in \mathcal{S}} \pi_i = 1.$$

The first set of equations may alternatively be written as

$$(\boldsymbol{\pi}\mathbf{P})^T = \mathbf{P}^T\boldsymbol{\pi}^T = \boldsymbol{\pi}^T.$$

From this it follows that $\boldsymbol{\pi}^T$ is an eigenvector of $\mathbf{P}^T$ with eigenvalue $\lambda = 1$.

Remark

There may sometimes be multiple (non-parallel) eigenvectors associated to the eigenvalue 1. However, this can only occur when the Markov chain has more than one recurrent class.

Stationary probabilities: Example 1

Assume that $\{X_n\}$ has state space $\mathcal{S} = \{0, 1, 2\}$, and transition matrix:

$$\mathbf{P} = \begin{pmatrix} 0.50 & 0.50 & 0.00 \\ 0.25 & 0.50 & 0.25 \\ 0.00 & 0.50 & 0.50 \end{pmatrix}$$

The transpose of **P** has eigenvalues satisfying the equation

$$\det \begin{pmatrix} 0.50 - \lambda & 0.25 & 0.00 \\ 0.50 & 0.50 - \lambda & 0.50 \\ 0.00 & 0.25 & 0.50 - \lambda \end{pmatrix} = 0$$

From this it is easy to show that the eigenvalues must satisfy:

$$\lambda(\lambda - 0.5)(\lambda - 1) = 0$$

yielding the eigenvalues: $\lambda = 0$, $\lambda = 0.5$ and $\lambda = 1$.

Stationary probabilities

In order to find an eigenvector $\mathbf{x}$ associated to the eigenvalue λ, we must solve the linear equations:

$$\mathbf{P}^T\mathbf{x} = \lambda\mathbf{x}.$$

By letting $\lambda = 1$, we get that:

$$\mathbf{x}^T = c \cdot (1, 2, 1)$$

where c is any non-zero constant.

Since we want the resulting vector to have components with sum equal to 1, we let $c = (1 + 2 + 1)^{-1} = \frac{1}{4}$.

Hence, we get the following stationary distribution:

$$\pi_0 = \frac{1}{4}, \quad \pi_1 = \frac{1}{2}, \quad \pi_2 = \frac{1}{4}.$$

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STK2130 – Modellering av stokastiske prosesser