UiO : Department of Mathematics
University of Oslo

Lecture 3. First passage probabilities

STK2130 – Modellering av stokastiske prosesser

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Brush up: Markov chains

Definition

A discrete time, discrete space stochastic process $\{X_0, X_1, X_2, ...\}$ is called a *time homogeneous Markov chain* if its *transition probabilities* satisfy the following:

$$\mathbb{P}(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, ..., X_0 = i_0) = \mathbb{P}(X_{n+1} = j \mid X_n = i)$$

for all states $i_0, i_1, \ldots, i_{n-1}, i, j \in \mathcal{S}$ and all $n \geq 0$.

Notation

- $P_{i,j} := \mathbb{P}(X_{n+1} = j \mid X_n = i) = \mathbb{P}(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, ..., X_0 = i_0)$.
- One-step transition probability matrix: $\mathbf{P} = [P_{i,j}]_{i,j\in\mathcal{S}}$, where $P_{i,j} \geq 0$ for all $i, j \in \mathcal{S}$, and $\sum_{j\in\mathcal{S}} P_{i,j} = 1$ for all $i \in \mathcal{S}$.

Brush up: Chapman–Kolmogorov equations

For a homogeneous Markov chain $\{X_n\}$, we denote

$$P^n_{i,j} := \mathbb{P}(X_{k+n} = j \mid X_k = i)$$

– probability that, starting from the state i, the process will get to the state j in n steps.

Chapman-Kolmogorov equations

$$P^{n+m}_{i,j} = \sum_{l \in \mathcal{S}} P^n_{i,l} P^m_{l,j}, \quad n, m \geq 1, \quad i, j \in \mathcal{S}.$$

Chapman-Kolmogorov equations in matrix form

Denote $\mathbf{P}^{(n)} := [P^n_{i,j}]_{i,j \in \mathcal{S}}$ – the matrix of n-step transition probabilities.

$$\mathbf{P}^{(n+m)} = \mathbf{P}^{(n)} \mathbf{P}^{(m)}.$$

First passage probabilities

Consider a Markov chain $\{X_n\}$ with state space $\mathcal{S}$ and transition probability matrix **P**, and let $\mathcal{A}$ be a non-empty proper subset of $\mathcal{S}$.

We want to calculate the following probability:

$$\beta := \mathbb{P}(X_k \in \mathcal{A} \text{ for some } 1 \le k \le m \mid X_0 = i),$$

where we assume that $i \notin \mathcal{A}$. In other words, we are interested in probability that $\{X\}$ will visit $\mathcal{A}$ at least once within m steps given that is starts in i.

In order to analyze this, we introduce:

$$N := \min\{n > 0 : X_n \in \mathcal{A}\}.$$

Thus, N represents the first time the Markov chain enters $\mathcal{A}$.

First passage probabilities

$$\beta := \mathbb{P}(X_k \in \mathcal{A} \text{ for some } 1 \le k \le m \mid X_0 = i).$$

Consider a new Markov chain $\{W_n\}$ derived from $\{X_n\}$ as follows:

$$W_n := \begin{cases} X_n, & \text{if } n < N, \\ A, & \text{if } n \ge N, \end{cases}$$

where A is an arbitrary state from $\mathcal{A}$.

Then the state space of $\{W_n\}$ becomes $(\mathcal{S} \setminus \mathcal{A}) \cup \{A\}$ and observe that, when $\{X_n\}$ enters $\mathcal{A}$, i.e. at (random!) time N, $\{W_n\}$ is absorbed in state A.

First passage probabilities: Transition probabilities of $\{W_n\}$

$$\beta := \mathbb{P}(X_k \in \mathcal{A} \text{ for some } 1 \le k \le m \mid X_0 = i), \quad W_n := \begin{cases} X_n, & \text{if } n < N, \\ A, & \text{if } n \ge N. \end{cases}$$

The transition probability matrix of $\{W_n\}$, denoted $\mathbf{Q} = (Q_{i,j})$, is given by:

- if $i \notin \mathcal{A}$, $j \notin \mathcal{A}$,
$$Q_{i,j} = P_{i,j}.$$
- if $i \notin \mathcal{A}$,
$$Q_{i,A} = \sum_{j \in \mathcal{A}} P_{i,j}.$$
- $Q_{A,A} = 1$.

We now have:

$$\begin{aligned} \beta &= \mathbb{P}(X_k \in \mathcal{A} \text{ for some } 1 \le k \le m \mid X_0 = i) \\ &= \mathbb{P}(W_m = A \mid W_0 = i) = Q^m_{i,A}. \end{aligned}$$

Example 4.12

In a sequence of independent flips of a fair coin, we introduce a Markov chain $\{X_n\}$ such that X_n represents *the length of the current sequence of consecutive heads*.

The state space of this Markov chain is $\mathcal{S} = \{0, 1, 2, ...\}$, and we note that $X_0 = 0$. Moreover, the transition probability matrix of $\{X_n\}$ is given by:

$$\mathbf{P} = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & \cdots \\ 1/2 & 0 & 1/2 & 0 & \cdots \\ 1/2 & 0 & 0 & 1/2 & \cdots \\ 1/2 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Problem

Let N denote the number of flips until there is a run of three consecutive heads. Find

$$\mathbb{P}(N \leq 8), \quad \mathbb{P}(N = 8).$$

Example 4.12

We then let $\mathcal{A} = \{3, 4, 5, ...\}$ and introduce:

$$N = \min\{n : X_n \in \mathcal{A}\}.$$

N denotes the number of flips until there is a run of 3 consecutive heads.

We want to calculate $\mathbb{P}(N \leq 8)$ and $\mathbb{P}(N = 8)$. To solve this problem, we introduce a Markov chain $\{W_n\}$ with states $\mathcal{T} = \{0, 1, 2, 3\}$ defined relative to the sequence of coin flips as follows:

$$W_n := \begin{cases} X_n & \text{if } n < N, \\ 3 & \text{if } n \geq N. \end{cases}$$

We note that state 3 is an absorbing state for W_n, and that the transition probability matrix of $\{W_n\}$ is given by:

$$\mathbf{Q} = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Example 4.12

We then calculate the 7 and 8 steps transition probability matrices:

$$\mathbf{Q}^{(7)} = \begin{pmatrix} 0.3438 & 0.1875 & 0.1016 & 0.3672 \\ 0.2891 & 0.1563 & 0.0859 & 0.4688 \\ 0.1875 & 0.1016 & 0.0547 & 0.6563 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{pmatrix}, \quad \mathbf{Q}^{(8)} = \begin{pmatrix} 0.3164 & 0.1719 & 0.0938 & 0.4180 \\ 0.2656 & 0.1445 & 0.0781 & 0.5117 \\ 0.1719 & 0.0938 & 0.0508 & 0.6836 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{pmatrix}$$

Using these matrices we get that:

$$\mathbb{P}(N \leq 8) = Q^8_{0,3} = 0.4180,$$
$$\mathbb{P}(N = 8) = \mathbb{P}(N \leq 8) - \mathbb{P}(N \leq 7) = Q^8_{0,3} - Q^7_{0,3} = 0.4180 - 0.3672 = 0.0508.$$

Alternatively:

$$\mathbb{P}(N = 8) = 0.5Q^7_{0,2} = 0.0508.$$

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