4.20) A transition prob. matrix is doubly stochastic if the sum of each column is one;

$$\sum_i P_{ij} = 1 \quad \forall j$$

If such a chain is irreducible and consists of $M+1$ states $0, 1, \dots, M$, show that the long run proportions are:

$$\pi_j = \frac{1}{M+1}, \quad j = 0, 1, \dots, M$$

oo

The Markov chain is finite state (since # states = $M+1$) and irreducible. Hence, it is positive recurrent (see Remark (ii) pg. 217).

Otherwise, all long run proportions are 0: Not possible for finite # states

Hence, from Thm. 4.1, the long run proportions are the unique solution of:

$$(\star)\quad \begin{cases} \pi_j = \sum_i \pi_i P_{ij}, & j \geq 1 \\ \sum_j \pi_j = 1 \end{cases}$$

Note that for $\pi_j = \frac{1}{M+1}$, $j = 0, \dots, M$, we have

$$\sum_i \pi_i P_{i,j} = \sum_i \frac{1}{M+1} P_{i,j} = \frac{1}{M+1} \sum_i P_{i,j} = \frac{1}{M+1} = \pi_j$$

doubly stochastic

Hence, $\pi_j = \frac{1}{M+1}$, $j = 0, \dots, M$

is the unique solution of $(\star)$ for our Markov chain, i.e. these are <u>the</u> long-run proportions.