chapter 10 exercises

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Solutions

10.2 Conditional distribution of standard brownian motion

Consider the expression for the joint density of standard brownian motion, equation (10.3) in the book. Since all increments are independent we can write the joint density of $f_{t_1}(a)$, $f_s(x)$ and $f_{t_2}(b)$ as

$$f(a, x, b) = f_{t_1}(a) f_{s-t_1}(x-a) f_{t_2-s}(b-x)$$

We know that the conditional density of a variable can be calculated from

$$f_{X|Y,Z}(x|y,z) = \frac{f_{X,Y,Z}(x,y,z)}{f_{Y,Z}(y,z)}$$

so plugging our joint density into this formula we get

$$f_{s|t_1,t_2}(x|A,B) = \frac{f_{t_1}(A) f_{s-t_1}(x-A) f_{t_2-s}(B-x)}{f_{t_1,t_2}(A,B)}$$

We can expand the denominator $f_{t_1,t_2}(A,B) = f_{t_2|t_1}(B|A) f_{t_1}(A) = f_{t_2-t_1}(B-A) f_{t_1}(A)$ and we get

$$f_{s|t_1,t_2}(x|A,B) = \frac{f_{s-t_1}(x-A) f_{t_2-s}(B-x)}{f_{t_2-t_1}(B-A)}$$

If we say that $t_1, A = 0$ we see that this expression matches the expression on the bottom of page 641

10.4 limits of hitting time

We know that

$$P(T_a \leq t) = \frac{2}{\sqrt{2\pi}} \int_{a/\sqrt{t}}^{\infty} e^{-y^2/2} dy$$

so we have

$$\begin{aligned} P(T_a \leq \infty) &= \lim_{t \to \infty} \frac{2}{\sqrt{2\pi}} \int_{a/\sqrt{t}}^{\infty} e^{-y^2/2} dy \\ &= 2 \int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \\ &= 2\frac{1}{2} \\ &= 1 \end{aligned}$$

Will update when I find an adequate solution for $E[T_a]$

10.9 joint density of brownian motion with drift

We have that the joint density of two variables is

$$f_{s,t}(x_1, x_2) = f_{t|s}(x_2|x_1) f_s(x_1)$$

since the increments are independent we have that

$$f_{t,s}(x_1, x_2) = f_{t-s}(x_2 - x_1) f_s(x_1)$$

which is trivial to find an exact expression for if you are interested.