Exercises chapter 7

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Solutions

7.1 Inequalities of $N(t)$ and S_n

Check if the following relations are true

a) $N(t) < n \iff S_n > t$?

This is trivially true from equation (7.2) in the book.

b) $N(t) \leq n \iff S_n \geq t$?

Let us first check if $N(t) \leq n \implies S_n \geq t$. Then we must have that $N(t) < n \implies S_n \geq t$, which we know is true from a), and $N(t) = n \implies S_n \geq t$. We know that $N(t) = n \implies S_n > t$ is not true, so we must check if $N(t) = n \implies S_n = t$. Let the interarrival times, $X_1, X_2, ...$, be any continuous random variables, then S_n is also a continuous random variable and $P(S_n = t) = 0$, see section 2.3 of the book to demonstrate this, so this relation is not true.

c) $N(t) > n \iff S_n < t$?

Let us first check if $S_n < t \implies N(t) > n$, which implies that $P(N(t) = n|S_n < t) = 0$. We know that $P(N(t) = n|S_n < t) = P(S_n \leq t|S_n < t) - P(S_{n+1} \leq t|S_n < t)$ and $P(S_n \leq t|S_n < t) = 1$, so we must have that $P(S_{n+1} \leq t|S_n < t) = 1$. Let again the interarrival times, $X_1, X_2, ...$ be any continuous random variables, then $P(S_{n+1} \leq t|S_n < t) = P(0 < X_{n+1} \leq 0) = P(X_{n+1} = 0) = 0$, so this statement is false.

7.2 Distribution of S_n and $N(t)$ when interrarival times are poisson distributed with mean μ

a)

We know that $S_n = \sum_{i=1}^{n} X_i,\ n \geq 1$ and that the sum of poisson-distributed random variables is poisson-distributed itself with mean as the sum of the means for all summands, so $S_n \sim Poisson(n\mu)$.

b)

We know that $P(S_n \leq t) = e^{-n\mu} \sum_{i=1}^{\lfloor t \rfloor} \frac{(n\mu)^j}{j!}$, so $P(N(t) = n) = P(S_n \leq t) - P(S_{n+1} \leq t) = e^{-n\mu} \sum_{i=1}^{\lfloor t \rfloor} \frac{(n\mu)^j}{j!} - e^{-(n+1)\mu} \sum_{i=1}^{\lfloor t \rfloor} \frac{((n+1)\mu)^j}{j!}$, where $\lfloor t \rfloor$ is t rounded down to the closest integer.