løsninger 23.02

Eivind Tørstad

February 2023

4.56

This is a variant of the gamblers ruin problem presented in section 4.5.1 of Ross. The solution is easily adapted from 4.5.1 by realizing we can define the states $i = 0, 1, ..., m + n$, where being in state $i = 0$ is equivalent to being down m, being in state $i = m + n$ is equivalent to being up n and being in state $i = m$ is equivalent to break-even.

Then let P_i denote the probability that, starting in state i, the gambler will eventually win(i.e. reach state $i = m+n$, i.e. be up n). From 4.5.1 the solution for P_i is:

$$P_i = \begin{cases} \frac{1-(\frac{1-p}{p})^i}{1-(\frac{1-p}{p})^{m+n}}, \text{ if } p \neq 1/2 \\ \frac{i}{m+n}, \text{ if } p = 1/2 \end{cases}$$

We want to calculate P_m, i.e. the probability going up n when starting at break-even. The solution then is

$$P_m = \begin{cases} \frac{1-(\frac{1-p}{p})^m}{1-(\frac{1-p}{p})^{m+n}}, \text{ if } p \neq 1/2 \\ \frac{m}{m+n}, \text{ if } p = 1/2 \end{cases}$$

4.57

This is also a variant of the gamblers ruin problem in 4.5.1. The solution is easily adapted from 4.5.1 by realizing that conditioned on the whether first step is clockwise or anti-clockwise, the probability of passing through all other states before reaching state 0 is the same as the probability of reaching the step before winning (i.e. reaching state $n - 1$) in the gamblers ruin problem.

Let $P_1^{(a)}$ denote the probability of passing through all other states before reaching state 0, conditioned on the first step being counter-clockwise. Then from section 4.5.1 we have

$$P_1^{(a)} = \begin{cases} \frac{1-(q/p)}{1-(q/p)^{n-1}}, \text{ if } p \neq 1/2 \\ \frac{1}{n-1}, \text{ if } p = 1/2 \end{cases}$$

Let $P_1^{(b)}$ be defined similarly, but conditioned on the first step being clockwise. Then we have

$$P_1^{(b)} = \begin{cases} \frac{1-(p/q)}{1-(p/q)^{n-1}}, \text{ if } p \neq 1/2 \\ \frac{1}{n-1}, \text{ if } p = 1/2 \end{cases}$$

Finally, let P_0 be the probability of passing through all other states before reaching state 0, conditioned on starting in state 0, the solution of our problem. Then by the law of total probability we have the solution $P_0 = pP_1^{(a)} + qP_1^{(b)}$

4.63

We have that

$$\mathbf{P}_T = \begin{pmatrix} 0.4 & 0.2 & 0.1 \\ 0.1 & 0.5 & 0.2 \\ 0.3 & 0.4 & 0.2 \end{pmatrix}$$

And from section 4.6, $\mathbf{S} = (\mathbf{I} - \mathbf{P}_T)^{-1}$. Solving in matlab gives

$$\mathbf{S} = \begin{pmatrix} 64/29 & 40/29 & 18/29 \\ 28/29 & 90/29 & 26/29 \\ 38/29 & 60/29 & 56/29 \end{pmatrix}$$

so $(s_{13}, s_{23}, s_{33}) = (18/29, 26/29, 56/29)$.

From 4.6 we have $f_{ij} = \frac{s_{ij} - \delta_{ij}}{s_{jj}}$ so $(f_{13}, f_{23}, f_{33}) = (9/28, 13/28, 27/56)$