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Based on some of the previous solutions provided by Anton Yurchenko-Tytarenko and Kristina Rognlien Dahl

Excercise 4.9

Define a markov-chain X_n with states $i = 0, 1, 2, 3$ where $i < 3$ is the number of consecutive heads and $i = 3$ is the absorbing state where 3 consecutive heads have been reached. The probability transition matrix for X_n is

$$\mathbf{P} = \begin{pmatrix} 0.4 & 0.6 & 0 & 0 \\ 0.4 & 0 & 0.6 & 0 \\ 0.4 & 0 & 0 & 0.6 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then the probability of getting three consecutive heads for $n \leq 10$ is $\mathbf{P}^{10}_{0,3} = 0.7$

Excercise 4.13

For this problem it will be enough to prove that $\mathbf{P}^{r+1}$ has all positive entries (since then by induction $\mathbf{P}^{r+n}$ must have all positive entries). Consider any $i, j \in \mathcal{S}$, then $\mathbf{P}^{r+1}_{i,j} = \sum_{s \in \mathcal{S}} \mathbf{P}^r_{i,s} \mathbf{P}_{s,j} = 0$ if and only if all the entries of the j-th column of $\mathbf{P}$ are 0. (Since $\mathbf{P}^r_{i,s} > 0$ for all $s \in \mathcal{S}$, then $\mathbf{P}_{s,j}$ must be 0 for all $s \in \mathcal{S}$ for $\mathbf{P}^{r+1}_{i,j}$ to be 0). But if the j-th column of $\mathbf{P}$ is all 0, then the probability of reaching state j from any state $s \in \mathcal{S}$ in $n \geq 1$ steps is also 0. Then it is impossible that $\mathbf{P}^r$ has all positive entries and we have arrived at a contradiction. Thus $\mathbf{P}$ cannot have a zero j-th column and $\mathbf{P}^{r+1}_{i,j} > 0$

Excercise 4.15

We let $\mathcal{N} := \{n \geq 0 | \mathbf{P}^n_{i,j} > 0\}$. We know that, since $i \to j$, $\mathcal{N} \neq \emptyset$, so we can consider $M^* := \min \mathcal{N}$. Assume for contradiction that $M^* > M$. By definition of M^* we have $\mathbf{P}^{M^*}_{i,j} > 0$. So there exists at path from

i to j, in M^* steps, where all the transitions have probability greater than 0. Denote the states of this path as $i_1, i_2, ..., i_{M^*-1}$ and let $i = i_0$. Then $\mathbf{P}_{i,j}^{M^*} = \mathbf{P}_{i_0,i_1}\mathbf{P}_{i_1,i_2}\cdots\mathbf{P}_{i_{M^*-1},j} > 0$. Since $M^* > M$, there must exist $n, m \in \{0, 1, ..., M^* - 1\}$, where $m < n$, such that $i_n = i_m$ (as in, since we are taking more steps than the number of states in the markov chain, we must be visiting at least one state twice). By the markov assumption $\mathbf{P}_{i_m,i_{n+1}} = \mathbf{P}_{i_n,i_{n+1}}$. So $\mathbf{P}_{i,j}^{M^*-n+m} = \mathbf{P}_{i_0,i_1}\mathbf{P}_{i_1,i_2}\cdots\mathbf{P}_{i_{m-1},i_m}\mathbf{P}_{i_m,i_{n+1}}\cdots\mathbf{P}_{i_{M^*-1},j} > 0$. Then $M^* - n + m \in \mathcal{N}$, but $M^* - n + m < M^*$ which condracticts the assumption $M^* = \min \mathcal{N}$. Therefore we must have $M^* \leq M$.

Excercise 4.17

We have a sequence of independent bernoulli trials where for $i = 1, 2, ..., n$

$$Y_i = \begin{cases} 1, & \text{with probability } p \\ -1, & \text{with probability } 1 - p \end{cases}$$

And a markov chain where at time n the state is $\sum_{i=1}^n Y_i$. The strong law of large numbers states that $\frac{1}{n}\sum_{i=1}^n Y_i \xrightarrow{a.s.} \mathrm{E}[Y]$ as $n \to \infty$. We have that $\mathrm{E}[Y] = -1(1-p) + 1p = 2p - 1$ which is obviously negative when $p < 1/2$, positive when $p > 1/2$ and zero when $p = 1/2$. Then $\sum_{i=1}^n Y_i \xrightarrow{a.s.} n(2p-1)$ as $n \to \infty$, where $n(2p-1)$ obviously diverges when $p \neq 1/2$. We have then that when $p \neq 1/2$, then $\Pr(\lim_{n\to\infty}\sum_{i=1}^n Y_i = \pm\infty) = 1$ (Definition of almost sure convergence). So there is a zero probability that the chain will visit any state infinitely often and the chain is transient.