Løsninger stk2130 16.02 V1

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Based on some of the previous solutions provided by Anton Yurchenko-Tytarenko and Kristina Rognlien Dahl

Excercise 4.18

a) We have two states $S = \{1, 2\}$, where $X_n = 1$ means we are flipping coin 1 and vice-versa. X_n has probability transition matrix

$$\mathbf{P} = \begin{pmatrix} 0.6 & 0.4 \\ 0.5 & 0.5 \end{pmatrix}$$

Since $\mathbf{P}$ only has one class (i.e. all states communicate) it is recurrent, and since it is both recurrent and finite-state, then by remark (ii) on page 217 X_n is positive recurrent. Then theorem 4.1 applies and we can find the stationary probabilities by solving the equations $(\pi_1, \pi_2) = (\pi_1, \pi_2)\mathbf{P}$ and $\pi_1 + \pi_2 = 1$.

We have $\pi_1 = 0.6\pi_1 + 0.5\pi_2$ and $\pi_1 = 0.4\pi_1 + 0.5\pi_2$. Solving for π_1 gives $\pi_1 = \frac{5}{9}$ which is what we are intersted in.

b) We are intersted in the probability that the fifth flip is with coin 2, when we start with flipping coin 1. In terms of our markov chain that means that $X_0 = 1$ and we are looking for the probability of transitioning to state two in the fourth transition s.t. the fifth transition is made from state 2. That means we are looking for $\mathbf{P}^4_{1,2}$. Which by matlab gives $\mathbf{P}^4_{1,2} = 0.444$.

c) We are interested in the proportion of flips that land heads. We define a variable Y

$$Y_n = \begin{cases} 1, & \text{when the n-th flip lands heads} \\ 0, & \text{otherwise} \end{cases}$$

We have then that $E(Y_n)$ is the proportion of flips that land heads. We have $E(Y_n) = E(Y_n|X_n = 1)P(X_n = 1) + E(Y_n|X_n = 2)P(X_n = 2)$ by the law of total expectation. Then $E(Y_n) = (1 * 0.6 + 0 * 0.4)\pi_1 + (1 * 0.5 + 0 * 0.5)\pi_2 = \frac{5}{9}$

Excercise 4.13

For this problem it will be enough to prove that $\mathbf{P}^{r+1}$ has all positive entries (since then by induction $\mathbf{P}^{r+n}$ must have all positive entries). Consider any $i, j \in \mathcal{S}$, then $\mathbf{P}^{r+1}_{i,j} = \sum_{s \in \mathcal{S}} \mathbf{P}^{r}_{i,s} \mathbf{P}_{s,j} = 0$ if and only if all the entries of the j-th column of $\mathbf{P}$ are 0. (Since $\mathbf{P}^{r}_{i,s} > 0$ for all $s \in \mathcal{S}$, then $\mathbf{P}_{s,j}$ must be 0 for all $s \in \mathcal{S}$ for $\mathbf{P}^{r+1}_{i,j}$ to be 0). But if the j-th column of $\mathbf{P}$ is all 0, then the probability of reaching state j from any state $s \in \mathcal{S}$ in $n \geq 1$ steps is also 0. Then it is impossible that $\mathbf{P}^{r}$ has all positive entries and we have arrived at a contradiction. Thus $\mathbf{P}$ cannot have a zero j-th column and $\mathbf{P}^{r+1}_{i,j} > 0$

Excercise 4.15

We let $\mathcal{N} := \{n \geq 0 | \mathbf{P}^{n}_{i,j} > 0\}$. We know that, since $i \to j$, $\mathcal{N} \neq \emptyset$, so we can consider $M^* := \min \mathcal{N}$. Assume for contradiction that $M^* > M$. By definition of M^* we have $\mathbf{P}^{M^*}_{i,j} > 0$. So there exists at path from i to j, in M^* steps, where all the transitions have probability greater than 0. Denote the states of this path as $i_1, i_2, ..., i_{M^*-1}$ and let $i = i_0$. Then $\mathbf{P}^{M^*}_{i,j} = \mathbf{P}_{i_0,i_1} \mathbf{P}_{i_1,i_2} \cdots \mathbf{P}_{i_{M^*-1},j} > 0$. Since $M^* > M$, there must exist $n, m \in \{0, 1, ..., M^* - 1\}$, where $m < n$, such that $i_n = i_m$ (as in, since we are taking more steps than the number of states in the markov chain, we must be visiting at least one state twice). By the markov assumption $\mathbf{P}_{i_m,i_{n+1}} = \mathbf{P}_{i_n,i_{n+1}}$. So $\mathbf{P}^{M^*-n+m}_{i,j} = \mathbf{P}_{i_0,i_1} \mathbf{P}_{i_1,i_2} \cdots \mathbf{P}_{i_{m-1},i_m} \mathbf{P}_{i_m,i_{n+1}} \cdots \mathbf{P}_{i_{M^*-1},j} > 0$. Then $M^* - n + m \in \mathcal{N}$, but $M^* - n + m < M^*$ which condracticts the assumption $M^* = \min \mathcal{N}$. Therefore we must have $M^* \leq M$.

Excercise 4.17

We have a sequence of independent bernoulli trials where for $i = 1, 2, ..., n$

$$Y_i = \begin{cases} 1, & \text{with probability } p \\ -1, & \text{with probability } 1 - p \end{cases}$$

And a markov chain where at time n the state is $\sum_{i=1}^{n} Y_i$. The strong law of large numbers states that $\frac{1}{n} \sum_{i=1}^{n} Y_i \xrightarrow{a.s.} \mathrm{E}[Y]$ as $n \to \infty$. We have that $\mathrm{E}[Y] = -1(1-p) + 1p = 2p - 1$ which is obviously negative when $p < 1/2$, positive when $p > 1/2$ and zero when $p = 1/2$. Then $\sum_{i=1}^{n} Y_i \xrightarrow{a.s.} n(2p-1)$ as $n \to \infty$, where $n(2p-1)$ obviously diverges when $p \neq 1/2$. We have then that when $p \neq 1/2$, then $\Pr(\lim_{n \to \infty} \sum_{i=1}^{n} Y_i = \pm\infty) = 1$ (Definition of almost sure convergence). So there is a zero probability that the chain will visit any state infinitely often and the chain is transient.