4.59

Let $n_k :=$ steps until state 0 or N is reached after step k.

Want: $E[n_0 | X_0 = i]$

Condition on X_1, use double expectation:

$E[E[n_0 | X_1 = j] | X_0 = i]$

$= E[1 + \underbrace{E[n_1 | X_1 = j]}_{= M_j} | X_0 = i]$

$= 1 + E[n_1 | X_1 = i+1] \cdot P(X_1 = i+1 | X_0 = i)$
$\quad + E[n_1 | X_1 = i-1] \cdot P(X_1 = i-1 | X_0 = i)$

$= 1 + pM_{i+1} + q M_{i-1}$

Solving equations:

$M_i = pM_i + qM_i = 1 + pM_{i+1} + qM_{i-1}$

$$\Rightarrow M_{i+1} - M_i = \frac{q}{p}(M_i - M_{i-1}) - \frac{1}{p}$$

Then,

$$M_2 - M_1 = \left(\frac{q}{p}\right)(M_1 - M_0) - \frac{1}{p} = \left(\frac{q}{p}\right)M_1 - \frac{1}{p}$$

$$M_3 - M_2 = \frac{q}{p}(M_2 - M_1) - \frac{1}{p} = \left(\frac{q}{p}\right)^2 M_1 - \frac{1}{p}\left(\frac{q}{p} + 1\right)$$

$$\vdots$$

$$M_N - M_{N-1} = \frac{q}{p}(M_{N-1} - M_{N-2}) - \frac{1}{p} = \left(\frac{q}{p}\right)^{N-1} M_1 - \frac{1}{p}\sum_{j=0}^{N-2}\left(\frac{q}{p}\right)^j$$

Sum the first i equations:

$$M_i - M_1 = M_i - M_{i-1} + M_{i-1} - M_{i-2} + \cdots$$

$$= \left(\frac{q}{p}\right)^{i-1} M_1 - \frac{1}{p}\sum_{j=0}^{i-2}\left(\frac{q}{p}\right)^j + \left(\frac{q}{p}\right)^{i-2} M_1 - \frac{1}{p}\sum_{j=0}^{i-3}\left(\frac{q}{p}\right)^j + \cdots$$

$$\sum_{k=1}^{i-2}\sum_{j=0}^{k}\left(\frac{q}{p}\right)^j = \underbrace{\left(\left(\frac{q}{p}\right)^0 + \left(\frac{q}{p}\right)^0 + \cdots + \left(\frac{q}{p}\right)^0\right)}_{i-1} + \underbrace{\left(\left(\frac{q}{p}\right)^1 + \cdots + \left(\frac{q}{p}\right)^1\right)}_{i-2} + \cdots$$

$$= \sum_{j=0}^{i-2}(i-1-j)\left(\frac{q}{p}\right)^j$$

$$= \left(\frac{q}{p}\right)^{i-1} M_1 - \frac{1}{p}\sum_{j=0}^{i-2}(i-1-j)\left(\frac{q}{p}\right)^j$$

$$= \left(\frac{q}{p}\right)^{i-1} M_1 - \frac{i}{p}\sum_{j=0}^{i-2}\left(\frac{q}{p}\right)^j + \frac{1}{p}\sum_{j=0}^{i-2}(j+1)\left(\frac{q}{p}\right)^j$$

Suppose $p = \frac{1}{2}$:

$$\sum_{j=0}^{i-2} \left(\frac{q}{p}\right)^j = i-1, \quad \sum_{j=0}^{i-2} (j+1)\left(\frac{q}{p}\right)^j = \sum_{j=1}^{i-1} j = \frac{i(i-1)}{2}$$

Now,

$$M_i = M_1 \sum_{j=0}^{i-1} \left(\frac{q}{p}\right)^j - \frac{i}{p} \sum_{j=0}^{i-2} \left(\frac{q}{p}\right)^j + \frac{1}{p} \sum_{j=0}^{i-2} (j+1) \left(\frac{q}{p}\right)^j$$

$$= i M_1 - \frac{i}{1/2}(i-1) + \frac{1}{1/2} \frac{i(i-1)}{2}$$

$$= i M_1 - 2i^2 + 2i + i^2 - i$$

$$= i(M_1 - i + 1)$$

Solve M_1:

$$0 = M_N = N(M_1 - N + 1) \Rightarrow M_1 = N - 1$$

$$M_i = i(N - 1 - i + 1) = i(N - i) \quad \blacksquare$$

Let $p \neq \frac{1}{2}$. Solve: $pM_{i+1} - M_i + (1-p) M_{i-1} = -1$

Homogenous equation:

$$pM_{i+1} - M_i + (1-p)M_{i-1} = 0$$

express as $px^2 - x + 1 - p = 0 \Rightarrow x_1 = 1 \quad x_2 = \frac{q}{p}$

Solution on form: $A + B\left(\frac{q}{p}\right)^i$, $A, B \in \mathbb{R}$

Particular solution: $M_i = Ci$:

$$pC(i+1) - Ci + qC(i-1) = -1$$

$$\Rightarrow C = \frac{1}{q-p}$$

$$\Rightarrow M_i = A + B\left(\frac{q}{p}\right)^i + \frac{i}{q-p}$$

Find A, B using $M_0 = M_N = 0$

$$0 = M_0 = A + B\left(\frac{q}{p}\right)^0 + 0 \Rightarrow A = -B$$

$$0 = M_N = A + B\left(\frac{q}{p}\right)^N + \frac{N}{q-p} \Rightarrow A = -\frac{N}{(q-p)\left(1-\left(\frac{q}{p}\right)^N\right)}$$

$$B = \frac{N}{(q-p)\left(1-\left(\frac{q}{p}\right)^N\right)}$$

$$\Rightarrow M_i = -\frac{N}{(q-p)\left(1-\left(\frac{q}{p}\right)^N\right)} + \frac{N}{(q-p)\left(1-\left(\frac{q}{p}\right)^N\right)}\left(\frac{q}{p}\right)^i + \frac{i}{q-p}$$

$$= \frac{i}{q-p} - \frac{N}{q-p} \cdot \frac{1-\left(\frac{q}{p}\right)^i}{1-\left(\frac{q}{p}\right)^N} \quad \blacksquare$$

4.61

a) Condition on first game:

$$P_i = \alpha_i P_{i+1} + (1-\alpha_i) P_{i-1}, \quad P_0 = 0, \quad P_N = 1$$

b) $$\alpha_i P_i + (1-\alpha_i) P_i = \alpha_i P_{i+1} + (1-\alpha_i) P_{i-1}$$

$$\alpha_i (P_{i+1} - P_i) = (1-\alpha_i)(P_i - P_{i-1})$$

$$P_{i+1} - P_i = \frac{1-\alpha_i}{\alpha_i}(P_i - P_{i-1})$$

$$P_2 - P_1 = \frac{1-\alpha_1}{\alpha_1}(P_1 - 0) = \frac{1-\alpha_1}{\alpha_1} P_1$$

$$P_3 - P_2 = \frac{1-\alpha_2}{\alpha_2}\left(\frac{1-\alpha_1}{\alpha_1} P_1\right)$$

$$\vdots$$

$$P_N - P_{N-1} = \prod_{j=1}^{N-1} \frac{1-\alpha_j}{\alpha_j} \cdot P_1$$

Sum i first equations:

$$P_i - P_1 = \sum_{k=1}^{i-1} \prod_{j=1}^{k} \frac{1-\alpha_j}{a_j} \cdot P_1$$

$$P_i = \sum_{k=0}^{i-1} \prod_{j=1}^{k} \frac{1-\alpha_j}{\alpha_j} \cdot P_1 \qquad \left(\prod_{j=1}^{0} x_j = 1 \right)$$

Solve P_1:

$$1 = P_N = P_1 \sum_{k=0}^{N-1} \prod_{j=0}^{k} \frac{1-\alpha_j}{\alpha_j}$$

$$\Rightarrow \quad P_1 = \frac{1}{\sum_{k=0}^{N-1} \prod_{j=0}^{k} \frac{1-\alpha_j}{\alpha_j}}$$

Set in:

$$P_i = \frac{\sum_{k=0}^{i-1} \prod_{j=1}^{k} \frac{1-\alpha_j}{\alpha_j}}{\sum_{k=0}^{N-1} \prod_{j=1}^{k} \frac{1-\alpha_j}{\alpha_j}} \qquad \blacksquare$$

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c) This is a gambler's ruin problem because there are two recurrent classes: 0 balls in urn 1, N balls in urn 1.