STK2130
Spring 21
Exercise sessions

Ch. 6: Continuous time Markov chains

Today: 6.5, 6.6

6.5) N individuals in a population, some infected:
Contact follows Poisson process rate λ, equally likely to involve any of the $\binom{N}{2}$ pairs of individuals. Contact between infected & non-infected $\Rightarrow$ Prob. p of non-infected becoming sick. An infected cannot become non-infected.

$X(t) = \#$ infected members of population at time t

a) Is $\{X(t), t \geq 0\}$ a cont. time Markov chain?

To answer this, assume $X(t) = n$.
We are interested in the transitions, i.e.,

$$X(t) = n \longrightarrow X(t+h) = n+1.$$

Note that this is the only possible transition, because an infected cannot become non-infected.

What is the intensity parameter of this transition?

The transition happens when contact between infected & non-infected. The probability of such a contact is:

$$\frac{n\,(N-n)}{\binom{N}{2}} \quad \left(\frac{\{\#\text{ infected}\}\cdot\{\#\text{ non-infected}\}}{\#\text{ possible pairs of contacts}} \right)$$

Probability = "gunstige" / mulige

Since contact happens with intensity λ, this type of contact happens with intensity:

$$\frac{\lambda\, n\,(N-n)}{\binom{N}{2}}$$

When this contact happens, the probability of transition (i.e., another infected individual), is p. Hence, the intensity of transition is:

$$\frac{p\lambda\, n\,(N-n)}{\binom{N}{2}}$$

a) $\{X(t), t \geq 0\}$ is a continuous time Markov chain because only the current number of infected individuals is used to calculate the probability of future transitions (from the intensity calculation above; only involves $n = X(t)$).

b) Specify its type.

$\{X(t), t \geq 0\}$ is a pure birth process, because we can only go from $X(t) = n$ to $X(t+h) = n+1$, that is; we can only add to $X(t)$.

If we instead consider $N - X(t)$, the number of non-infected, we get a pure death process (since can only go from n to $n-1$).

c) Starting with a single infected individual, what is the expected time until all members are infected?

We would like to find $E[S_{N-1}]$, where

$$S_{N-1} := \sum_{i=1}^{N-1} \underbrace{T_i}_{\text{time between "arrivals", i.e. new infections}}$$

is the time where the remaining $N-1$ individuals have been infected.

$T_i :=$ time between when there are i infected & when there are $i+1$ infected individuals

From the calculations above, the intensity of new infections is

$$\frac{p\lambda(N-n)}{\binom{N}{2}}.$$

Hence, the time between new infections,

$$T_i \sim \exp\left(\frac{p\lambda i(N-i)}{\binom{N}{2}}\right)$$

since the times between arrivals in Poisson processes are exponentially distributed.

Hence,

$$E[S_{N-1}] = E\left[\sum_{i=1}^{N-1} T_i\right]$$

$$= \sum_{i=1}^{N-1} E[T_i]$$

$$= \sum_{i=1}^{N-1} \frac{\binom{N}{2}}{\lambda p \, i(N-i)} = \frac{1}{\lambda p} \binom{N}{2} \sum_{i=1}^{N-1} \frac{1}{i(N-i)}$$

Expectation of exp. with param. λ is $\frac{1}{\lambda}$

Expected time until all individuals are infected

6.6.) Birth & death process; Birth rates $\lambda_i = (i+1)\lambda$, $i \geq 0$ and death rates $\mu_i = i\mu$, $i \geq 0$.

a) Expected time to go from state 0 to 4:

See also derivation pg. 381-382 in book for methodology!

Let

$$I_i := \begin{cases} 1 & \text{if the first transition from } i \text{ is to } i+1 \\ 0 & \text{if} \text{ ---''--- } i-1 \end{cases}$$

Let $T_i :=$ time it takes to go from i to $i+1$.

Then,

indep. of whether first trans. is birth or death, time until transition is exp. rate $\lambda_i + \mu_i$

$$E[T_i \mid I_i = 1] = \frac{1}{\lambda_i + \mu_i}$$

(6.3) in book

$$E[T_i \mid I_i = 0] = \underbrace{\frac{1}{\lambda_i + \mu_i}}_{\text{same arg. as above}} + \underbrace{E[T_{i-1}]}_{\text{time to go from } i-1 \text{ to } i} + \underbrace{E[T_i]}_{\text{time to go from } i \text{ to } i+1}$$

If first transition is to $i+1$, do not need to add more time

Hence,

$$E[T_i] = E[T_i \mid I_i = 1]\, P(I_i = 1) + E[T_i \mid I_i = 0] \cdot P(I_i = 0)$$

law of total expectation

$$= \frac{1}{\lambda_i + \mu_i} \frac{\lambda_i}{\lambda_i + \mu_i} + \left(\frac{1}{\lambda_i + \mu_i} + E[T_{i-1}] + E[T_i] \right) \cdot \frac{\mu_i}{\lambda_i + \mu_i}$$

$P(I_i = 1)$
$= P(\text{birth happens bef. death})$
$= \frac{\lambda_i}{\lambda_i + \mu_i}$
since both times between births & deaths are exponentially distributed.

Similarly,
$P(I_i = 0) = P(\text{death bef. birth})$
$= \frac{\mu_i}{\lambda_i + \mu_i}$

$$= \frac{\lambda_i}{(\lambda_i + \mu_i)^2} + \frac{\mu_i}{(\lambda_i + \mu_i)^2} + \frac{\mu_i}{\lambda_i + \mu_i} \left(E[T_{i-1}] + E[T_i] \right)$$

$\Downarrow$ (Solve for $E[T_i]$)

$$(*) : E[T_i] = \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i} E[T_{i-1}]$$

Also, note that

$$E[T_0] = \frac{1}{\lambda_0} \quad : (**)$$

time to go from 0 to 1; only births can happen
$\Rightarrow$ Exp. time until first birth $= \frac{1}{\lambda_0}$
since time until birth is exponential

Therefore, based on ($\star$) & ($\star\star$), we can iteratively compute any $E[T_i]$.

This can be used in a) & b):

a)
$$E\begin{bmatrix}\text{time to go from} \\ 0 \text{ to } 4\end{bmatrix} = \overbrace{E[T_0]}^{\text{go from 0 to 1}} + \overbrace{E[T_1]}^{\text{from 1 to 2}} + \overbrace{E[T_2]}^{\text{from 2 to 3}} + \underbrace{E[T_3]}_{\text{from 3 to 4}}$$

where,

$$\underset{(\star)}{E[T_1]} = \frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1}\underbrace{\frac{1}{\lambda_0}}_{E[T_0]}$$

$$E[T_2] = \frac{1}{\lambda_2} + \frac{\mu_2}{\lambda_2}\left(\frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1}\frac{1}{\lambda_0}\right)$$

$$E[T_3] = \frac{1}{\lambda_3} + \frac{\mu_3}{\lambda_3}\left(\frac{1}{\lambda_2} + \frac{\mu_2}{\lambda_2}\left(\frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1}\frac{1}{\lambda_0}\right)\right).$$

What is the expected time from 2 to 5?

b) Similarly,

$$E\begin{bmatrix}\text{time to go from} \\ 2 \text{ to } 5\end{bmatrix} = \overbrace{E[T_2]}^{\text{go from 2 to 3}} + \overbrace{E[T_3]}^{\text{from 3 to 4}} + \overbrace{E[T_4]}^{\text{from 4 to 5}}$$

where

$$E[T_4] = \frac{1}{\lambda_4} + \frac{\mu_4}{\lambda_2}\left(\frac{1}{\lambda_3} + \frac{\mu_3}{\lambda_3}\left(\frac{1}{\lambda_2} + \frac{\mu_2}{\lambda_2}\left(\frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1}\frac{1}{\lambda_0}\right)\right)\right)$$

c) What are the variances in a) & b)?

Note that (6.3) is equivalent to

$$(\Delta) \quad E[T_i \mid I_i] = \frac{1}{\lambda_i + \mu_i} + (1 - I_i)\left(E[T_{i-1}] + E[T_i]\right)$$

From the def. of I_i

and that

$I_i \sim$ Bernoulli with success probability $p = \frac{\lambda_i}{\lambda_i + \mu_i}$ (i.e., the prob. of a birth happening before a death).

Then,

From (Δ) & computation rules for variance

$$(\mathrm{I}) \quad Var(E[T_i \mid I_i]) = Var(I_i)\left(E[T_{i-1}] + E[T_i]\right)^2$$

$$= \frac{\lambda_i}{\lambda_i + \mu_i} \frac{\mu_i}{\lambda_i + \mu_i}\left(E[T_{i-1}] + E[T_i]\right)^2$$

$I_i \sim$ Bernoulli, $p = \frac{\lambda_i}{\lambda_i + \mu_i}$

Where, $X_i \sim$ exp with mean $\frac{1}{\lambda_i + \mu_i}$, since given $I_i = 1$, time to reach $i+1$ is expec. time for transition from i

$$(\mathrm{II}) \quad Var(T_i \mid I_i = 1) = Var(X_i \mid I_i = 1)$$

the time until transition i indep. of next state

$$= Var(X_i)$$

See pg. 383 in book!

$$= \frac{1}{(\lambda_i + \mu_i)^2} \quad (6.5)$$

$X_i \sim$ exp with mean $\frac{1}{\lambda_i + \mu_i}$

$$\mathrm{Var}(T_i \mid I_i = 0) = \mathrm{Var}\Big(X_i + \text{time to go from } i-1 \text{ to } i + \text{time to go from } i \text{ to } i+1\Big)$$

See pg. 383 in Ross?

$$= \mathrm{Var}(X_i) + \mathrm{Var}(T_{i-1}) + \mathrm{Var}(T_i) \quad (6.6)$$

Independent; Var(sum) = sum (Var)

X_i, T_{i-1}, T_i are indep.

def of I_i

$$\mathrm{Var}(T_i \mid I_i) = \mathrm{Var}(X_i) + (1 - I_i)\big[\mathrm{Var}(T_{i-1}) + \mathrm{Var}(T_i)\big]$$

by def's

So,

$$E[\mathrm{Var}(T_i \mid I_i)] = \frac{1}{(\lambda_i + \mu_i)^2} + \frac{\mu_i}{\mu_i + \lambda_i}\big(\mathrm{Var}(T_{i-1}) + \mathrm{Var}(T_i)\big)$$

(6.7) in book

From conditional variance formula, we find (see pg. 384) that

$$\mathrm{Var}(T_i) = \frac{1}{\lambda_i(\lambda_i + \mu_i)} + \frac{\mu_i}{\lambda_i}\mathrm{Var}(T_{i-1}) + \frac{\mu_i}{\mu_i + \lambda_i}\big(E[T_{i-1}] + E[T_i]\big)^2 \quad (\star\star\star)$$

Since $\mathrm{Var}(T_0) = \frac{1}{\lambda_0^2}$ (T_0 is time from 0 to 1 $\rightsquigarrow$ only births can happen & time until birth is exponential with param. λ_0), we can

use (★★★) and formula from a) for $E[T_i]$
to iteratively compute $Var(T_i)$.

In a)

$$Var\left(\begin{matrix}\text{time go}\\ \text{from 0 to 4}\end{matrix}\right) = Var(T_0 + T_1 + T_2 + T_3)$$

$$= \underbrace{Var(T_0) + \cdots + Var(T_3)}_{\text{Find this via iterative use of (★★★)}}$$

The T_i's are independent, so variance of sum is sum of variances

In b):

$$Var\left(\begin{matrix}\text{time go}\\ \text{from 2 to 5}\end{matrix}\right) = Var(T_2 + T_3 + T_4)$$

$$= \underbrace{Var(T_2) + Var(T_3) + Var(T_4)}_{\text{Find this iteratively via (★★★)}}$$

$=$