

Ex: 4.16, 4.35, 4.52, 4.60

4.16.) Show that if i is recurrent and i does not communicate with j, then Pij = 0.

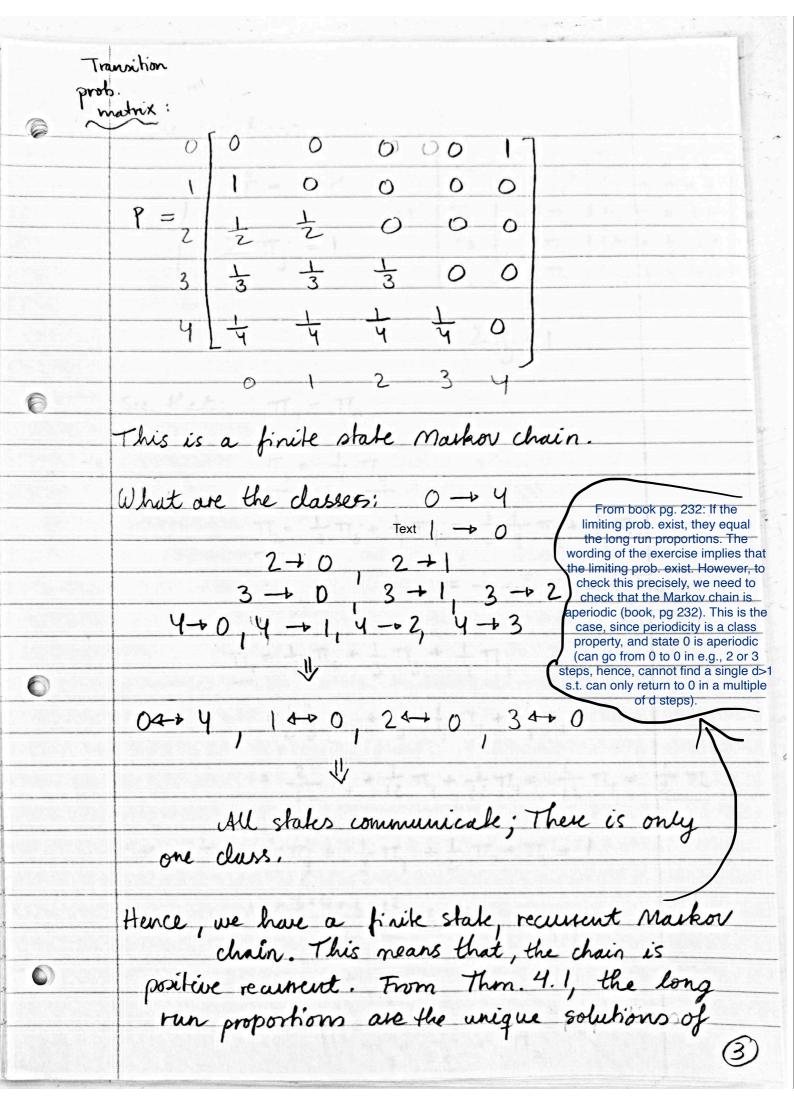
Interpretation: Once a process enters a recurrent class, it can never leave. Therefore, recurrent classes are sometimes called closed.

Assume, for contradiction, Pi, > 0 (i recurrent, i does not communicate with j).

Then,  $P_{j,i}^{(n)} = 0 \, \forall \, n$ , because otherwise (if  $P_{j,i}^{(n)} > 0$  for some n, so we can go from j to i at some time) i and j would communicate (which we have assumed that they don't).

This means that the process, starting in i, has a positive prob. of dt least Pi, of never returning to i (because you can go from i to j in just one skep with prob. Pi,).

This contradicts the assumption that i is recurrent (recurrence means we return to i with prob. 1). Hence, our assumption is fulse, so Pij = 0. 4.35) markov chain, states 0, 1, 2, 3, 4. Po, 4 = 1, When in i, i>0, the next state is equally likely to be any of 0,1,..., i-1. Find the lineiting probabilities.  $P_{0,0} = P_{0,1} = P_{0,2} = P_{0,3} = 0$  (since prob. sum to 1 8  $P_{0,4} = 1$ )  $P_{1,0} = 1$ ,  $P_{1,1} = P_{1,2}$ ,  $P_{1,3}$ ,  $P_{1,4} = 0$  (-u - 2 equally likely assumption)  $P_{3,0} = \frac{1}{3} = P_{3,1} = P_{3,2} + P_{3,3} = P_{3,4} = 0 \quad (-\pi - )$   $P_{4,0} = \frac{1}{4} = P_{4,1} = P_{4,2} = P_{4,3} + P_{4,3} = 0 \quad (-\pi - )$ 



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$$\frac{T_3}{3} = \frac{1}{4} \frac{T_4}{4}$$

$$=\frac{1}{4}\frac{4}{3}\pi_{4}=\frac{1}{3}\pi_{4}$$

$$\Pi_1 = \frac{1}{2} \Pi_2 + \frac{1}{3} \Pi_3 + \frac{1}{4} \Pi_4$$

$$= \frac{1}{2} \frac{1}{3} \pi_4 + \frac{1}{3} \cdot \frac{1}{4} \pi_4 + \frac{1}{4} \pi_4$$

$$= \frac{2}{12} \pi_{y} + \frac{1}{12} \pi_{y} + \frac{3}{12} \pi_{y} = \frac{6}{12} \pi_{y} = \frac{1}{2} \pi_{y}$$

$$\frac{12+3+4+6+12}{12} + \pi_{4} = 1$$

$$\frac{37}{12}\pi_{4} = 1$$

$$T_4 = \frac{12}{37} = T_6$$

$$T_1 = \frac{1}{2} t_4 = \frac{1}{2} \frac{12}{37} = \frac{6}{37}$$

$$T_2 = \frac{1}{3}T_4 = \frac{1}{3}\frac{12}{37} = \frac{4}{37}$$

$$T_3 = \frac{1}{4} T_4 = \frac{1}{4} \frac{12}{37} = \frac{3}{37}$$

Hunce, the long run proportions are

$$\vec{7} = \left[ \begin{array}{cccc} \frac{12}{37} & \frac{6}{37} & \frac{4}{37} & \frac{3}{37} & \frac{12}{37} \end{array} \right]$$

4.52.) Taxi driver drives in two zones. Fares from A have destination A with prop. 0, 6 & in B with prob. 0, 4. Faxes from B have destination A with prob. 0,3 & B with prob. 0,7.

Elpwhit | trip in A ] = 6

Espropit | trip in B] = 8 Espropit | trin in A&B] = 12

Find the aug. profit per trip.

Set S = { A, B}. The process {Xn In 20 of places the text driver goes is a Markov chain because the rext location only depends on the aurent distribution of the location.

$$P = \begin{cases} 0,6 & 0,4 \\ 0,3 & 0,7 \end{cases}$$

Set  $(profit)_n$  denote the driver's profit from nice n. Let  $(nide)_n := (X_{n-1}, X_n)$  initial distination location We are interested in lin E[(profit)n] E[(profit)n]= E[E[(profit)n](nide)n]]  $= E[E[(prohit)_n | (X_{n-1}, X_n)]]$ of expectation =  $\sum_{i,j \in S \times S} E[(profit)_n | (X_{n-1}, X_n) = (i,j)]$ .  $P((\chi_{n-1},\chi_n)=(i,j))$  $= \sum_{i,j \in S \times S} E[(pwhit)_n | (X_{n-i}, X_n) = (i,j)]$ · P(Xn=j/Xn-1=i) P(Xn-1=i) on the state of Kn-,; to say more

$$= E[(pwht)_{n} | (X_{n-1}, X_{n}) = (A_{1}A_{1})] P_{A_{1}A_{1}} P(X_{n-1} = A_{1})$$

$$= A_{1}A_{1} P_{A_{1}A_{1}} P(X_{n-1} = A_{1}) P_{A_{1}A_{1}} P(X_{n-1} = A_{1})$$

$$= A_{1}A_{1} P_{A_{1}A_{1}} P(X_{n-1}, X_{n}) = (A_{1}B_{1})] P_{A_{1}B_{1}} P(X_{n-1} = B_{1})$$

$$+ E[(pwht)_{n} | (X_{n-1}, X_{n}) = (B_{1}B_{1})] P_{B_{1}A_{1}} P(X_{n-1} = B_{1})$$

$$+ E[(pwht)_{n} | (X_{n-1}, X_{n}) = (B_{1}B_{1})] P_{B_{1}A_{1}} P(X_{n-1} = B_{1})$$

$$= A_{1}A_{1} P(X_{n-1} = A_{1}) + 12 P_{A_{1}B_{1}} P(X_{n-1} = A_{1})$$

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$$\pi_{A} = 0, 6 \, \pi_{A} + 0, 3 \, \pi_{B}$$

$$= 0, 6 \, \pi_{A} + 0, 3 \, (1 - \pi_{A})$$

$$= 0, 6 \, \pi_{A} + 0, 3 \, (1 - \pi_{A})$$

$$\pi_{A} = 0, 3 \, \pi_{A} = 0, 3$$

$$0, 7 \, \pi_{A} = 0, 3$$

$$\pi_{A} = \frac{0, 3}{0, 7} = \frac{3}{7}$$

$$= \flat \, \pi_{B} = \frac{4}{7}$$

$$\text{Hence from (*):}$$

$$\lim_{n \to \infty} E[(proh!)_{n}] = 6 \cdot 0, 6 \cdot \frac{3}{7} + 12 \cdot 0, \frac{4}{7} \cdot \frac{3}{7}$$

$$+ 12 \cdot 0, 3 \cdot \frac{4}{7} + 8 \cdot 0, 7 \cdot \frac{4}{7}$$

$$\approx 8, 86$$
So the awage prohit per trip for the taxi denier is
$$8, 86$$

4.60) Markov chain, states 
$$S := \{1,2,3,4\}$$
.

$$P = \begin{cases} 0,4 & 0,3 & 0,2 & 0,1 \\ 0,2 & 0,2 & 0,2 & 0,4 \\ 3 & 0,25 & 0,25 & 0,5 & 0 \\ 4 & 0,2 & 0,1 & 0,4 & 0,3 \end{cases}$$
Assume  $X := 1$ .

Assume Xo = 1.

a) Find P (process enters state 3 before state 4).

Let p: = prob. enter plate 3 before 4 when starting in state i.

Then 
$$(from P)$$
:
$$(P_1 = 0, 4P_1 + 0, 3P_2 + 0, 2P_3 + 0, 1P_4 \quad (I)$$

$$\int P_2 = 0.2P_1 + 0.2P_2 + 0.2P_3 + 0.4P_4 \quad (II)$$

$$\int P_3 = 1$$

$$(1-\frac{1}{8})p_1 = \frac{1}{8} + \frac{1}{3} = \frac{11}{24}$$

$$\frac{7}{8} P_1 = \frac{11}{24}$$

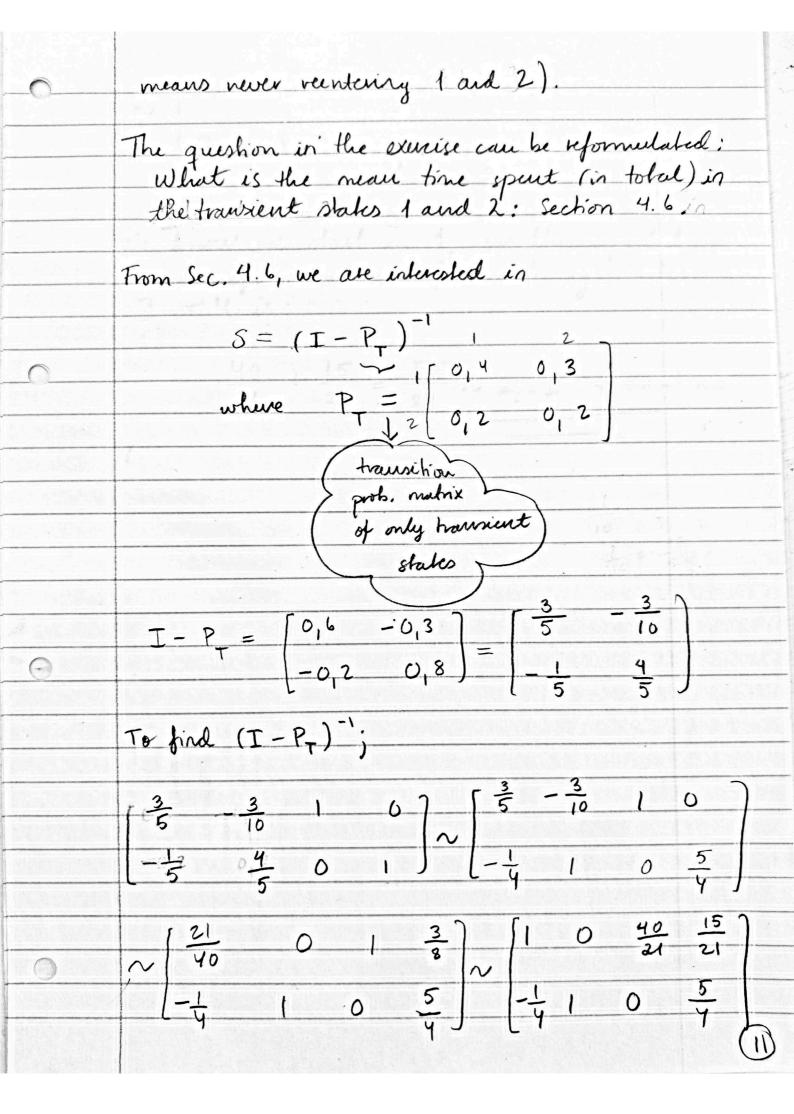
$$P_1 = \frac{11}{21}$$

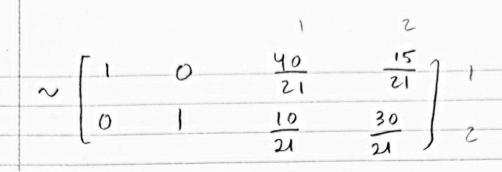
So the probability of entiring 3 before 4 when we know we short in 1 is  $\frac{11}{21}$ .

Consider the tweaked markov chain where states 3 & 4 are made absorbing, e.g.

$$\widetilde{P} = 
\begin{bmatrix}
0_1 & 0_1 & 0_1 & 0_1 & 0_1 \\
0_1 & 0_1 & 0_1 & 0_1 & 0_1 \\
0 & 0 & 0 & 0
\end{bmatrix}$$

Then, the dates I and I are transient (since there is a pos. prob. of entering 3 and 4, which





We know we start in 1, so the expected number of transitions before entering states 3 or 4 is set in

$$\frac{40}{21} + \frac{15}{21} = \frac{55}{21} \approx 2,62$$
 transitions