STK2130

Exercise session 3: Ch. 4 Markov chains

Ex: 4.16, 4.35, 4.52, 4.60

4.16) Show that if i is recurrent and i does not communicate with j, then $P_{i,j} = 0$.

Interpretation: Once a process enters a recurrent class, it can never leave. Therefore, recurrent classes are sometimes called closed.

Assume, for contradiction, $P_{i,j} > 0$ (i recurrent, i does not communicate with j).

Then, $P_{j,i}^{(n)} = 0 \ \forall n$, because otherwise (if $P_{j,i}^{(n)} > 0$ for some n, so we can go from j to i at some time) i and j would communicate (which we have assumed that they don't).

This means that the process, starting in i, has a positive prob. of at least $P_{i,j}$ of never returning to i (because you can go from i to j in just one step with prob. $P_{i,j}$).

This contradicts the assumption that i is recurrent (recurrence means we return to i with prob. 1).

Hence, our assumption is false, so $P_{i,j} = 0$. $\boxed{}$

4.35) Markov chain, states $0, 1, 2, 3, 4$. $P_{0,4} = 1$,

When in i, $i > 0$, the next state is equally likely to be any of $0, 1, \ldots, i-1$.

Find the limiting probabilities.

$P_{0,0} = P_{0,1} = P_{0,2} = P_{0,3} = 0$ (since prob. sum to 1 & $P_{0,4} = 1$)

$P_{1,0} = 1$, $P_{1,1} = P_{1,2}, P_{1,3}, P_{1,4} = 0$ (— " — & equally likely assumption)

$P_{2,0} = \frac{1}{2} = P_{2,1}$, $P_{2,2} = P_{2,3} = P_{2,4} = 0$ (— " —)

$P_{3,0} = \frac{1}{3} = P_{3,1} = P_{3,2}$, $P_{3,3} = P_{3,4} = 0$ (— " —)

$P_{4,0} = \frac{1}{4} = P_{4,1} = P_{4,2} = P_{4,3}$, $P_{4,4} = 0$ (— " —)

Transition prob. matrix:

$$P = \begin{array}{c|ccccc} & & & & & \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 3 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 4 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \\ \hline & 0 & 1 & 2 & 3 & 4 \end{array}$$

This is a finite state Markov chain.

What are the classes: $0 \to 4$

Text $1 \to 0$

$2 \to 0$, $2 \to 1$

$3 \to 0$, $3 \to 1$, $3 \to 2$

$4 \to 0$, $4 \to 1$, $4 \to 2$, $4 \to 3$

$\Downarrow$

$0 \leftrightarrow 4$, $1 \leftrightarrow 0$, $2 \leftrightarrow 0$, $3 \leftrightarrow 0$

$\Downarrow$

All states communicate; There is only one class.

From book pg. 232: If the limiting prob. exist, they equal the long run proportions. The wording of the exercise implies that the limiting prob. exist. However, to check this precisely, we need to check that the Markov chain is aperiodic (book, pg 232). This is the case, since periodicity is a class property, and state 0 is aperiodic (can go from 0 to 0 in e.g., 2 or 3 steps, hence, cannot find a single d>1 s.t. can only return to 0 in a multiple of d steps).

Hence, we have a finite state, recurrent Markov chain. This means that, the chain is positive recurrent. From Thm. 4.1, the long run proportions are the unique solutions of the equations

$$\begin{cases} \vec{\pi} = \vec{\pi} P \\ \sum_j \pi_j = 1 \end{cases} \Rightarrow \begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{bmatrix}^T = \begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \end{bmatrix}$$

$$\sum \pi_j = 1$$

See that: $\pi_4 = \pi_0$

$$\underline{\pi_3} = \underline{\frac{1}{4}\pi_4}$$

$$\underline{\pi_2} = \frac{1}{3}\pi_3 + \frac{1}{4}\pi_4 = \frac{1}{3}\cdot\frac{1}{4}\pi_4 + \frac{1}{4}\pi_4$$

$$= \frac{1}{\cancel{4}}\cdot\frac{\cancel{4}}{3}\pi_4 = \underline{\frac{1}{3}\pi_4}$$

$$\underline{\pi_1} = \frac{1}{2}\pi_2 + \frac{1}{3}\pi_3 + \frac{1}{4}\pi_4$$

$$= \frac{1}{2}\cdot\frac{1}{3}\pi_4 + \frac{1}{3}\cdot\frac{1}{4}\pi_4 + \frac{1}{4}\pi_4$$

$$= \frac{2}{12}\pi_4 + \frac{1}{12}\pi_4 + \frac{3}{12}\pi_4 = \frac{6}{12}\pi_4 = \underline{\frac{1}{2}\pi_4}$$

So: $\pi_4 + \frac{1}{4}\pi_4 + \frac{1}{3}\pi_4 + \frac{1}{2}\pi_4 + \pi_4 = 1$

$$\frac{12+3+4+6+12}{12}\pi_4 = 1$$

$$\frac{37}{12}\pi_4 = 1$$

$$\pi_4 = \underline{\frac{12}{37}} = \pi_0$$

$$\pi_1 = \frac{1}{2}\pi_4 = \frac{1}{2}\,\frac{12}{37} = \underline{\frac{6}{37}}$$

$$\pi_2 = \frac{1}{3}\pi_4 = \frac{1}{3}\,\frac{12}{37} = \underline{\frac{4}{37}}$$

$$\pi_3 = \frac{1}{4}\pi_4 = \frac{1}{4}\,\frac{12}{37} = \underline{\frac{3}{37}}$$

Hence, the long run proportions are

$$\vec{\pi} = \left[\frac{12}{37}, \frac{6}{37}, \frac{4}{37}, \frac{3}{37}, \frac{12}{37}\right].$$

4.52.) Taxi driver drives in two zones. Fares from A have destination A with prop. 0,6 & in B with prob. 0,4. Fares from B have destination A with prob. 0,3 & B with prob. 0,7.
$E[\text{profit} \mid \text{trip in } A] = 6$
$E[\text{profit} \mid \text{trip in } B] = 8$
$E[\text{profit} \mid \text{trip in } A \,\&\, B] = 12$
Find the avg. profit per trip.

Let $S = \{A, B\}$. The process $\{X_n\}_{n\geq 0}$ of places the taxi driver goes is a Markov chain because the next location only depends on the current distribution of the location.

$$P = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0,6 & 0,4 \\ 0,3 & 0,7 \end{bmatrix} \end{array}$$

Let $(\text{profit})_n$ denote the driver's profit from ride n. Let $(\text{ride})_n := (\underbrace{X_{n-1}}_{\text{initial location}}, \underbrace{X_n}_{\text{destination}})$.

So: $(\text{ride})_1 = (X_0, X_1)$

We are interested in $\lim_{n\to\infty} E[(\text{profit})_n]$.

$$E[(\text{profit})_n] \overset{\text{double expectation}}{=} E[\,E[(\text{profit})_n \mid (\text{ride})_n]\,]$$

$$\overset{\text{def. } (\text{ride})_n}{=} E[\,E[(\text{profit})_n \mid (X_{n-1}, X_n)]\,]$$

$$\overset{\text{def. of expectation}}{=} \sum_{i,j \in S\times S} E[(\text{profit})_n \mid (X_{n-1}, X_n) = (i,j)] \cdot P((X_{n-1}, X_n) = (i,j))$$

$$= \sum_{i,j\in S\times S} E[(\text{profit})_n \mid (X_{n-1}, X_n) = (i,j)] \cdot P(X_n = j \mid X_{n-1} = i)\, P(X_{n-1} = i)$$

condition on the state of X_{n-1}; need to know where we start to say more.

$$= E[(\text{profit})_n \mid (X_{n-1}, X_n) = (A,A)]\, P_{A,A}\, P(X_{n-1} = A)$$

Cover all combinations $(i,j) \in S \times S$

$$+ E[(\text{profit})_n \mid (X_{n-1}, X_n) = (A,B)]\, P_{A,B}\, P(X_{n-1} = A)$$

$$+ E[(\text{profit})_n \mid (X_{n-1}, X_n) = (B,A)]\, P_{B,A}\, P(X_{n-1} = B)$$

$$+ E[(\text{profit})_n \mid (X_{n-1}, X_n) = (B,B)]\, P_{B,B}\, P(X_{n-1} = B)$$

$$= 6 \cdot P_{A,A}\, P(X_{n-1} = A) + 12\, P_{A,B}\, P(X_{n-1} = A)$$

assumptions in text

$$+ 12\, P_{B,A}\, P(X_{n-1} = B) + 8\, P_{B,B}\, P(X_{n-1} = B)$$

$$\xrightarrow[n \to \infty]{} 6\, P_{A,A}\, \pi_A + 12\, P_{A,B}\, \pi_A + 12\, P_{B,A}\, \pi_B + 8\, P_{B,B}\, \pi_B := (\star)$$

Recall; are interested in $\lim_{n\to\infty} E[(\text{profit})_n]$

where $\vec{\pi} = (\pi_A, \pi_B)$ are the limiting distribution / long run proportion.

$\{X_n\}_{n \geq 0}$ is an irreducible, finite state Markov chain, hence it is positive recurrent.

From Thm. 4.1, we can find the long run proportions $\vec{\pi}$ by solving:

$$\begin{cases} \vec{\pi} = \vec{\pi} P \\ \sum_{j \in S} \pi_j = 1 \end{cases}$$

$$\begin{cases} (\pi_A, \pi_B) = (\pi_A, \pi_B) \begin{bmatrix} 0{,}6 & 0{,}4 \\ 0{,}3 & 0{,}7 \end{bmatrix} \\ \pi_B = 1 - \pi_A \end{cases}$$

$$\pi_A = 0{,}6\,\pi_A + 0{,}3\,\pi_B$$
$$= 0{,}6\,\pi_A + 0{,}3\,(1-\pi_A)$$
$$= 0{,}6\,\pi_A + 0{,}3 - 0{,}3\,\pi_A$$

$$\pi_A - 0{,}3\,\pi_A = 0{,}3$$

$$0{,}7\,\pi_A = 0{,}3$$

$$\pi_A = \frac{0{,}3}{0{,}7} = \underline{\frac{3}{7}}$$

$$\Rightarrow \pi_B = \underline{\frac{4}{7}}$$

Hence, from $(\star)$:

$$\lim_{n\to\infty} E[\,(\text{profit})_n\,] = 6\cdot 0{,}6\cdot \frac{3}{7} + 12\cdot 0{,}4\cdot \frac{3}{7} + 12\cdot 0{,}3\cdot \frac{4}{7} + 8\cdot 0{,}7\cdot \frac{4}{7}$$

$$\approx \underline{8{,}86}$$

So the average profit per trip for the taxi driver is $\underline{\underline{8{,}86}}$.

4.60) Markov chain, states $S := \{1,2,3,4\}$.

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0{,}4 & 0{,}3 & 0{,}2 & 0{,}1 \\ 0{,}2 & 0{,}2 & 0{,}2 & 0{,}4 \\ 0{,}25 & 0{,}25 & 0{,}5 & 0 \\ 0{,}2 & 0{,}1 & 0{,}4 & 0{,}3 \end{bmatrix} \end{array}$$

Assume $X_0 = 1$.

a) Find P(process enters state 3 before state 4).

Let $p_i :=$ prob. enter state 3 before 4 when starting in state i.

Then (from P):

$$\begin{cases} p_1 = 0{,}4\,p_1 + 0{,}3\,p_2 + 0{,}2\,p_3 + 0{,}1\,p_4 & \text{(I)} \\ p_2 = 0{,}2\,p_1 + 0{,}2\,p_2 + 0{,}2\,p_3 + 0{,}4\,p_4 & \text{(II)} \\ p_3 = 1 \\ p_4 = 0 \end{cases}$$

Solve this: From (I),

$$0{,}6\,p_1 = 0{,}3\,p_2 + 0{,}2$$
$$p_1 = \frac{1}{2}p_2 + \frac{1}{3}$$

From (II),

$$0{,}8\,p_2 = 0{,}2\,p_1 + 0{,}2$$
$$p_2 = \frac{1}{4}p_1 + \frac{1}{4}$$

Insert (II) in (I):

$$P_1 = \frac{1}{2}\left(\frac{1}{4}P_1 + \frac{1}{4}\right) + \frac{1}{3}$$

$$\left(1 - \frac{1}{8}\right)P_1 = \frac{1}{8} + \frac{1}{3} = \frac{11}{24}$$

$$\frac{7}{8}P_1 = \frac{11}{24}$$

$$\underline{P_1 = \frac{11}{21}}$$

So the probability of entering 3 before 4 when we know we start in 1 is $\underline{\underline{\frac{11}{21}}}$.

b) What is the mean number of transitions until either 3 or 4 is entered?

Consider the tweaked Markov chain where states 3 & 4 are made absorbing, e.g.

$$\tilde{P} = \begin{bmatrix} 0{,}4 & 0{,}3 & 0{,}2 & 0{,}1 \\ 0{,}2 & 0{,}2 & 0{,}2 & 0{,}4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then, the states 1 and 2 are transient (since there is a pos. prob. of entering 3 and 4, which means never reentering 1 and 2).

The question in the exercise can be reformulated: What is the mean time spent (in total) in the transient states 1 and 2: Section 4.6 in

From Sec. 4.6, we are interested in

$$S = (I - P_T)^{-1}$$

$$\text{where} \quad P_T = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 0{,}4 & 0{,}3 \\ 0{,}2 & 0{,}2 \end{bmatrix} \end{array}$$

transition prob. matrix of only transient states

$$I - P_T = \begin{bmatrix} 0{,}6 & -0{,}3 \\ -0{,}2 & 0{,}8 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & -\frac{3}{10} \\ -\frac{1}{5} & \frac{4}{5} \end{bmatrix}$$

To find $(I - P_T)^{-1}$;

$$\left[\begin{array}{cc|cc} \frac{3}{5} & -\frac{3}{10} & 1 & 0 \\ -\frac{1}{5} & \frac{4}{5} & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} \frac{3}{5} & -\frac{3}{10} & 1 & 0 \\ -\frac{1}{4} & 1 & 0 & \frac{5}{4} \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} \frac{21}{40} & 0 & 1 & \frac{3}{8} \\ -\frac{1}{4} & 1 & 0 & \frac{5}{4} \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & \frac{40}{21} & \frac{15}{21} \\ -\frac{1}{4} & 1 & 0 & \frac{5}{4} \end{array}\right]$$

$$\sim \begin{array}{cc} \begin{matrix} & & 1 & 2 \end{matrix} & \\ \begin{bmatrix} 1 & 0 & \frac{40}{21} & \frac{15}{21} \\ 0 & 1 & \frac{10}{21} & \frac{30}{21} \end{bmatrix} & \begin{matrix} 1 \\ 2 \end{matrix} \end{array}$$

We know we start in 1, so the expected number of transitions before entering states 3 or 4 is

$$\frac{40}{21} + \frac{15}{21} = \frac{55}{21} \approx 2{,}62 \text{ transitions}$$