Exercices ch. 5;

STK 2130,
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The exponential distribution

Today: 5.1, 5.2, 5.6, 5.8, 5.12,
Exam 2009 - ex. 1.

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5.1) Time T req. to repair a machine is exp. distributed with mean $\frac{1}{2}$.

a) $P(T > \frac{1}{2})$?

Since T is exp. distributed with mean

$\frac{1}{\lambda} = \frac{1}{2}$, we find that $\lambda = 2$.

So $T \sim \exp(2)$, hence

$$F(t) = \begin{cases} 1 - e^{-\lambda t}, & t \geq 0 \\ 0, & t < 0 \end{cases} = \begin{cases} 1 - e^{-2t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Sec. 5.2

$:= P(T \leq t)$

From this,

$$P(T > \frac{1}{2}) = 1 - P(T \leq \frac{1}{2}) = 1 - F(\frac{1}{2})$$
$$= 1 - (1 - e^{-2 \cdot \frac{1}{2}})$$
$$= e^{-1} = \underline{\underline{\frac{1}{e}}}$$

b) $P(T \geq 12{,}5 \mid T > 12)$?

$$P(T \geq 12{,}5 \mid T > 12) = P(T \geq \overbrace{12}^{t} + \overbrace{\tfrac{1}{2}}^{s} \mid T > \overbrace{12}^{t})$$

$$= P(T > \overbrace{\tfrac{1}{2}}^{s}) = \frac{1}{e}$$

a)

$T \sim \exp(2)$,
so T is memoryless
(see Sec. 5.2.2)

5.2) Arrive at single-teller bank; 5 others. One being served, 4 in queue. Get in queue. Assume service times $\sim \exp(\mu)$. What is

E[time in bank] ?

$$\text{time in bank} = \underbrace{\text{time to serve current customer}} + \sum_{i=2}^{5} \underbrace{\left(\text{time to serve cust. } i\right)}_{\sim \exp(\mu)} + \underbrace{\text{time to serve me}}_{\sim \exp(\mu)}$$

Since this customer is still being served at the time we enter, this is $\sim \exp(\mu)$ due to memorylessness of the exp. distribution

$$= \sum_{i=1}^{6} X_i$$

$X_i \sim \exp(\mu),\ i = 1, \dots, 6$

So,

$$E[\text{time in bank}] = E\left[\sum_{i=1}^{6} X_i\right]$$

$$= \sum_{i=1}^{6} E[X_i]$$

linearity of expectation

$$= \sum_{i=1}^{6} \frac{1}{\mu} = \frac{6}{\mu}$$

$X_i \sim \exp(\mu)$

$\Downarrow$

sec. 5.2

$E[X_i] = \frac{1}{\mu}$

5.6) Ex. 5.3; Post office, two clerks. S enters; J is served by one & B by the other.

S is told: Service starts when J or B leaves.

Time clerk i spends with customer $\sim \exp(\lambda_i)$, $i = 1, 2$.

Show: $P(S \text{ is not last}) = \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^2 + \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^2$

$$P(S \text{ not last}) = P(S < J \mid B < J)\, P(B < J)$$

$$+ P(S < B \mid J < B)\, P(J < B)$$

condition on who finishes first of J & B

law of total probability

$$= P(S < J \mid S \text{ with clerk } 2)\, P(B < J)$$

$$+ P(S < J \mid S \text{ with clerk } 1)\, P(J < B)$$

since S gets service as soon as one other leaves, the distribution of the remaining customer is still $\exp(\mu_i)$ at the time S gets service. This is due to memorylessness of the exp. distribution.

Note also that the prob. of J & B finishing at the same time is 0, since this is a continuous prob. distribution.

$$= \frac{\lambda_2}{(\lambda_1+\lambda_2)} \cdot \frac{\lambda_2}{(\lambda_1+\lambda_2)} + \frac{\lambda_1}{(\lambda_1+\lambda_2)} \cdot \frac{\lambda_1}{(\lambda_1+\lambda_2)}$$

J is with clerk 1, B is with clerk 2;

Recall $P(X_1 < X_2) = \frac{\lambda_1}{\lambda_1+\lambda_2}$ when $X_1, X_2 \sim \exp(\lambda_i)$ from Sec. 5.2.3

$$= \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^2 + \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^2$$

5.8) X, Y indep. exp. r.v. with rates λ & μ.

What is cond. distribution of X given $X < Y$?

$$P(X \mid X < Y) = P(\min\{X, Y\})$$

since $\min\{X, Y\} = X$ when we know $X < Y$

But from Proposition 5.2,

$$\min\{X, Y\} \sim \exp(\lambda + \mu)$$

since X, Y are indep., exp. r.V's.

Hence,

$$f_{X \mid X<Y}(x) = \begin{cases} (\lambda+\mu)e^{-(\lambda+\mu)x}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

From sec. 5.2; expressions for pdf. of exp. distr.

5.12) X_i, $i = 1, 2, 3$, indep., exp. r.v.s with rates λ_i, $i = 1, 2, 3$.

a) $P(X_1 < X_2 < X_3)$?

$$P(X_1 < X_2 < X_3) = P(\min\{X_1, X_2, X_3\} = X_1 \text{ and } \min\{X_2, X_3\} = X_2)$$

$$= P(\min\{X_2, X_3\} = X_2 \mid \min\{X_1, X_2, X_3\} = X_1) \cdot P(\min\{X_1, X_2, X_3\} = X_1)$$

def of cond. probability: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

$$= P(\min\{X_2, X_3\} = X_2)\, P(\min\{X_1, X_2, X_3\} = X_1)$$

independence; knowing X_1 is smallest gives no info about ranking of X_2 & X_3

$$= \frac{\lambda_2}{\lambda_2 + \lambda_3} \cdot \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3}$$

Sek. 5.2.3; $P(X_1 < X_2) = \dfrac{\lambda_1}{\lambda_1 + \lambda_2}$ when X_1, X_2 indep. exp. with rates λ_1, λ_2

Also, for 2nd factor, see bottom of pg. 305:

$$P(X_i = \min_j X_j) = P(X_i < \min_{j \neq i} X_j) = \frac{\lambda_i}{\sum_{j=1}^{n} \lambda_j}$$

b)

$$P\left(X_1 < X_2 \;\middle|\; \max_{1\le i\le 3}\{X_i\} = X_3\right)$$

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

def. conditional probability;

$X_1 < X_2$ & $\max_i X_i = X_3$

$\Updownarrow$

$X_1 < X_2 < X_3$

$$= \frac{P(X_1 < X_2 < X_3)}{P(\max_{1\le i\le 3}\{X_i\} = X_3)}$$

$\max X_i = X_3$ can only happen in two ways;

$X_1 < X_2 < X_3$ or $X_2 < X_1 < X_3$

$$= \frac{P(X_1 < X_2 < X_3)}{P(X_1 < X_2 < X_3) + P(X_2 < X_1 < X_3)}$$

$$= \frac{\cancel{\frac{\lambda_2}{\lambda_2+\lambda_3}} \cdot \cancel{\frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3}}}{\cancel{\frac{\lambda_2}{\lambda_2+\lambda_3}} \cdot \cancel{\frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3}} + \frac{\lambda_1}{\lambda_1+\lambda_3}\,\cancel{\frac{\lambda_2}{\lambda_1+\lambda_2+\lambda_3}}}$$

From a)

$$= \frac{\frac{1}{\lambda_2+\lambda_3}}{\frac{1}{\lambda_2+\lambda_3} + \frac{1}{\lambda_1+\lambda_3}}$$

$$=$$

5.12) c)

$$E[\max_{i \in \{1,2,3\}} X_i \mid X_1 < X_2 < X_3]$$

$$= E[X_3 \mid X_1 < X_2 < X_3]$$

see also arg. Ex. 5.8

$$= E[\underbrace{\text{time until first failure}}_{\sim \exp(\lambda_1+\lambda_2+\lambda_3)} + \underbrace{\text{additional time until second failure}}_{\sim \exp(\lambda_2+\lambda_3)} + \underbrace{\text{additional time until third failure}}_{\sim \exp(\lambda_3)}]$$

memorylessness

minimum of 3 exp. r.v's → minimum of X_2, X_3

$X_3 \mid X_1 < X_2 < X_3$ is the time where the third component fails, given that this is the last to fail. Hence, we wait for comp. 1 to fail ($\exp(\lambda_1+\lambda_2+\lambda_3)$) then comp. 2 to fail ($\exp(\lambda_2+\lambda_3)$), and finally for component 3 to fail ($\exp(\lambda_3)$)

This follows from memorylessness & independence of X_1, X_2, X_3

$$= \frac{1}{\lambda_1+\lambda_2+\lambda_3} + \frac{1}{\lambda_2+\lambda_3} + \frac{1}{\lambda_3}$$

d) Correction error; Insert expression above into old version of d) (in equality number two).

Consider a discrete-time Markov chain $\{X_n : n \geq 0\}$ with state space $\mathcal{X} = \{0, 1, 2, 3\}$, and transition probability matrix:

$$\boldsymbol{P} = \begin{bmatrix} p & q & 0 & 0 \\ 0 & 0 & p & q \\ 0 & q & 0 & p \\ 0 & p & q & 0 \end{bmatrix}$$

where $0 < p < 1$, $0 < q < 1$ and $p + q = 1$.

a) Describe the Markov chain by a diagram.

SOLUTION:

See Figure 1.

b) The chain has two classes, $\mathcal{C}_1 = \{0\}$ and $\mathcal{C}_2 = \{1, 2, 3\}$. For each of these classes discuss whether the class is *transient* or *recurrent.*

SOLUTION:

We consider the probabilities:

$$f_i = P\left(\bigcup_{r=1}^{\infty} \{X_r = i\} \mid X_0 = i\right), \quad i \in \mathcal{X}.$$

From the textbook we have that state i is *transient* if $f_i < 1$ and *recurrent* if $f_i = 1$.

In this case we have:

$$f_0 = P\left(\bigcup_{r=1}^{\infty} \{X_r = 0\} \mid X_0 = 0\right) = 1 - P\left(\bigcap_{r=1}^{\infty} \{X_r \neq 0\} \mid X_0 = 0\right)$$

$$= 1 - P(X_1 = 1 \mid X_0 = 0) = 1 - q < 1.$$

Hence, we conclude that $\mathcal{C}_1 = \{0\}$ is *transient.*

From the diagram it is easy to see that $i \leftrightarrow j$ for all $i, j \in \mathcal{C}_2$. Hence, these states belong to the same class. Since transience and recurrence are class properties, it follows that either all states in $\mathcal{C}_2$ are *transient* or all states in $\mathcal{C}_2$ are *recurrent.* However, the Markov chain has a finite state space, which implies that at least one state must be *recurrent.* Thus, the only possibility is that $\mathcal{C}_2 = \{1, 2, 3\}$ is *recurrent.*

c) Show that the two-step transition probability matrix is given by:

$$\boldsymbol{P}^{(2)} = \begin{bmatrix} p^2 & pq & pq & q^2 \\ 0 & 2pq & q^2 & p^2 \\ 0 & p^2 & 2pq & q^2 \\ 0 & q^2 & p^2 & 2pq \end{bmatrix}$$

Problem 1

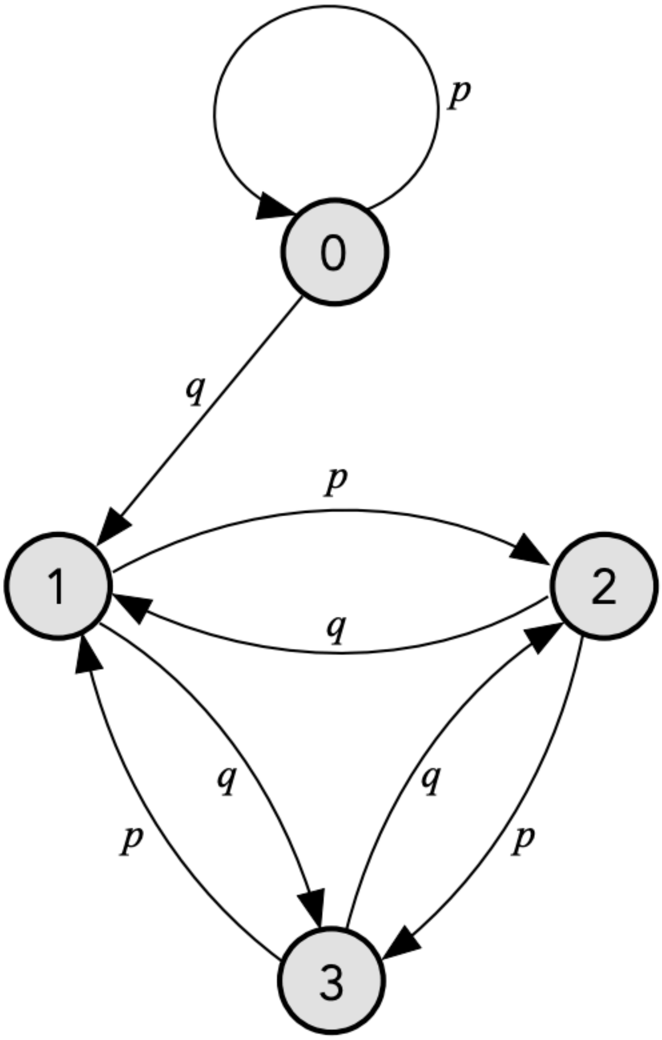

Figure 1: Diagram representing the Markov chain in Problem 1a

SOLUTION:

We have that:

$$\boldsymbol{P}^{(2)} = \boldsymbol{P} \cdot \boldsymbol{P} = \begin{bmatrix} p & q & 0 & 0 \\ 0 & 0 & p & q \\ 0 & q & 0 & p \\ 0 & p & q & 0 \end{bmatrix} \cdot \begin{bmatrix} p & q & 0 & 0 \\ 0 & 0 & p & q \\ 0 & q & 0 & p \\ 0 & p & q & 0 \end{bmatrix}$$

$$= \begin{bmatrix} p^2 & pq & pq & q^2 \\ 0 & 2pq & q^2 & p^2 \\ 0 & p^2 & 2pq & q^2 \\ 0 & q^2 & p^2 & 2pq \end{bmatrix}$$

In more detail:

$$P^2_{ij} = \sum_{k \in \mathcal{X}} P_{ik} \cdot P_{kj}, \qquad \text{for all } i, j \in \mathcal{X}.$$

Hence, we have:

$$P^2_{0,0} = P_{0,0}P_{0,0} + P_{0,1}P_{1,0} + \cdots + P_{0,3}P_{3,0} = p^2$$

$$P^2_{0,1} = P_{0,0}P_{0,1} + P_{0,1}P_{1,1} + \cdots + P_{0,3}P_{3,1} = pq$$

$$P^2_{0,2} = P_{0,0}P_{0,2} + P_{0,1}P_{1,2} + \cdots + P_{0,3}P_{3,2} = pq$$

$$P^2_{0,3} = P_{0,0}P_{0,3} + P_{0,1}P_{1,3} + \cdots + P_{0,3}P_{3,4} = q^2$$

$$\cdots\cdots$$

d) Conditioned upon that the chain has entered $\mathcal{C}_2$, find the stationary distribution over these three states.

SOLUTION:

We let:

$$\boldsymbol{Q} = \begin{bmatrix} P_{1,1} & P_{1,2} & P_{1,3} \\ P_{2,1} & P_{2,2} & P_{2,3} \\ P_{3,1} & P_{3,2} & P_{3,3} \end{bmatrix} = \begin{bmatrix} 0 & p & q \\ q & 0 & p \\ p & q & 0 \end{bmatrix}$$

denote the submatrix of $\boldsymbol{P}$ containing the transition probabilities for the recurrent states $1, 2, 3$. Furthermore, we let $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$ denote the stationary distribution over these states. Then $\boldsymbol{\pi}$ must satisfy $\pi_1 + \pi_2 + \pi_3 = 1$ and:

$$\boldsymbol{\pi}\boldsymbol{Q} = \boldsymbol{\pi}$$

From the last set of equations we get that:

$$\begin{aligned} q\pi_2 + p\pi_3 &= \pi_1 \\ p\pi_1 + q\pi_3 &= \pi_2 \end{aligned}$$

Since $q = 1 - p$, these equations can be written as:

$$\begin{aligned} (1-p)\pi_2 + p\pi_3 &= \pi_1 \\ p\pi_1 + (1-p)\pi_3 &= \pi_2 \end{aligned}$$

We then multiply the first equation by p, and rearrange the terms:

$$\begin{aligned} p^2\pi_3 &= p\pi_1 - p(1-p)\pi_2 \\ (1-p)\pi_3 &= -p\pi_1 + \pi_2 \end{aligned}$$

We then add the two equations and get:

$$(p^2 - p + 1)\pi_3 = (p^2 - p + 1)\pi_2$$

This implies that $\pi_2 = \pi_3$. By inserting this into e.g., the first equation, we get that:

$$q\pi_2 + p\pi_2 = \pi_1$$

This implies that $\pi_2 = \pi_1$. Thus, we conclude that $\pi_1 = \pi_2 = \pi_3$, and since also $\pi_1 + \pi_2 + \pi_3 = 1$, it follows that:

$$\pi_1 = \pi_2 = \pi_3 = \frac{1}{3}$$

Alternatively, the result that the stationary distribution is *uniform* follows directly by the fact that the matrix $\boldsymbol{Q}$ is *doubly stochastic*.