6.10) 2 machines. Machine $i$ operates exponential time with rate $\lambda_i$ and then fails. Its repair time is exponential with rate $\mu_i$, $i=1,2$. The machines act independently of one another. Def. a 4-state, cont. time Markov chain that describes the state of the two machines. Use independence to compute transition probabilities for the chain. Check that these satisfy the forward & backward Kolmogorov equations.

∞ <u>Set up cont. time M-C</u> :

4 states: $(0,0)$ ; both machines work

$(1,0)$; machine 1 is down, machine 2 work

$(0,1)$; machine 2 is down; machine 1 works

$(1,1)$; both machines are down

$$\begin{tikzcd}[column sep=huge, row sep=large]
(0,0) \arrow[r, bend left, "\lambda_1"] \arrow[d, bend right, "\lambda_2"'] & (1,0) \arrow[l, bend left, "\mu_1"] \arrow[d, bend right, "\lambda_2"'] \\
(0,1) \arrow[u, bend right, "\mu_2"'] \arrow[r, bend right, "\lambda_1"'] & (1,1) \arrow[u, bend right, "\mu_2"'] \arrow[l, bend right, "\mu_1"']
\end{tikzcd}$$

$\lambda_{(0,0),(1,0)} = \lambda_1$

$\lambda_{(0,0),(0,1)} = \lambda_2$

$\lambda_{(1,1),(0,1)} = \mu_1$

$\vdots$ (transition rates from diagram above)

Then,

$\underbrace{v_{0,0}} = \lambda_1 + \lambda_2$ ; since if both work, transitions happen by either 1 or 2 failing (with rates $\lambda_1$, $\lambda_2$ resp.)

Rate of time spent in state 0,0 (both work) bef. trans. to diff. state

$v_{0,1} = \lambda_1 + \mu_2$ ; since if 2 is down, 1 works, transitions happen by either 1 failing (rate $\lambda_1$) or 2 being fixed (rate $\mu_2$)

$v_{1,1} = \mu_1 + \mu_2$ ; since if both are down, transitions happen by either being fixed (rates $\mu_1$, $\mu_2$ resp.)

$v_{1,0} = \lambda_2 + \mu_1$ ; since if 1 is down, 2 works, transitions happen by either 1 being fixed (rate $\mu_1$) or 2 failing (rate $\lambda_2$).

## Compute transition probabilities

see pg. 376

By independence of the two machines,

$$(\star\star)\quad \underbrace{P_{(i,j),(k,e)}(t)}_{\substack{\text{total transition prob.;}\\ \text{prob. machines 1 \& 2 will be in}\\ \text{k, e resp. at time t when starting}\\ \text{in i, j (resp.)}}} = \underbrace{P^{(1)}_{i,k}(t)}_{\substack{\text{transition prob. machine 1;}\\ \text{Prob. machine 1 will be in}\\ \text{k at t when starting in i}}} \cdot \underbrace{P^{(2)}_{j,e}(t)}_{\substack{\text{transition prob. machine 2;}\\ \text{Prob. machine 2 will be in}\\ \text{e at time t, when starting in j.}}}$$

Then, for each machine, 1, 2, we are in the setting of Example 6.11. From this example, — middle pg. 392

$$P^{(1)}_{0,0}(t) = \frac{\mu_1}{\mu_1+\lambda_1} + \frac{\lambda_1}{\mu_1+\lambda_1} e^{-(\lambda_1+\mu_1)t}$$

$$P^{(1)}_{1,0}(t) = \frac{\mu_1}{\mu_1+\lambda_1} + \frac{\mu_1}{\mu_1+\lambda_1} e^{-(\lambda_1+\mu_1)t}$$

Because the problem is symmetric, by swiching the roles of $\lambda_1$ and $\mu_1$;

essentially switching the roles of functioning / failed & using the result of Example 6.11 on this new problem

$$P^{(1)}_{1,1}(t) = \frac{\lambda_1}{\mu_1+\lambda_1} + \frac{\mu_1}{\mu_1+\lambda_1} e^{-(\lambda_1+\mu_1)t}$$

$$P^{(1)}_{0,1}(t) = \frac{\lambda_1}{\mu_1+\lambda_1} - \frac{\lambda_1}{\mu_1+\lambda_1} e^{-(\lambda_1+\mu_1)t}$$

You can set up the same kinds of equations for $P^{(2)}_{i,j}$ for machine 2 by substituting $\lambda_2$ (for $\lambda_1$) and $\mu_2$ (for $\mu_1$).

From this and (★★), we find the transition probabilities $P_{(i,j),(k,e)}(t)$. For example,

$$P_{(0,0),(0,0)}(t) = P^{(1)}_{0,0}(t) \cdot P^{(2)}_{0,0}(t)$$

$$= \frac{1}{\lambda_1+\mu_1}\left(\mu_1 + \lambda_1 e^{-(\mu_1+\lambda_1)t}\right)$$

$$\cdot \frac{1}{\lambda_2+\mu_2}\left(\mu_2 + \lambda_2 e^{-(\mu_2+\lambda_2)}\right)$$

We check that the backwards Kolmogorov equation holds in this case. The forward equation can be checked similarly (do this yourself!):

Thm. 6.1

In general, Kolmogorov's backward eqns say that

Thm. 6.2

$$\underbrace{P'_{i,j}(t)}_{:=\text{LHS}} = \overbrace{\sum_{k\neq i} \underbrace{q_{i,k}}_{} P_{k,j}(t) - v_i P_{i,j}(t)}^{:=\text{RHS}}$$ for all states $i, j$ and times $t$.

instantaneous transition rates; $v_i = \sum_j q_{ij}$. $q_{ij}$ is the rate, when in state $i$, at which the process trasitions into state $j$

In principle, we should check this for all $i, j = 0, 1$. Since all are checked in the same way, we only check for $i = (0,0)$, $j = (0,0)$.

Try another one for practice!

$$\text{RHS} = -P_{(0,0),(0,0)}(t)(\lambda_1 + \lambda_2) + \lambda_2 P_{(0,1),(0,0)}(t) + \lambda_1 P_{(1,0),(0,0)}(t)$$

use first part of exercise

Possible ways to transition into $j=(0,0)$

$$= -P^{(1)}_{0,0}(t) P^{(2)}_{0,0}(t)(\lambda_1 + \lambda_2) + \lambda_2 P^{(1)}_{0,0}(t) P^{(2)}_{1,0}(t) + \lambda_1 P^{(1)}_{1,0}(t) P^{(2)}_{0,0}(t)$$

use (**)

$$= -\frac{\lambda_1+\lambda_2}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)}\left(\mu_1+\lambda_1 e^{-(\lambda_1+\mu_1)t}\right) \cdot \left(\mu_2+\lambda_2 e^{-(\lambda_2+\mu_2)t}\right)$$

Insert expressions for $P_{i,j}^{(n)}(t)$, $i,j=0,1$, $n=1,2$

$$+\frac{\lambda_2}{\lambda_1+\mu_1}\left(\mu_1+\lambda_1 e^{-(\lambda_1+\mu_1)t}\right)\frac{1}{\lambda_2+\mu_2} \cdot \left(\mu_2-\mu_2 e^{-(\lambda_2+\mu_2)t}\right)$$

$$+\frac{\lambda_1}{\lambda_1+\mu_1}\left(\mu_1-\mu_1 e^{-(\lambda_1+\mu_1)t}\right)\frac{1}{\lambda_2+\mu_2} \cdot \left(\mu_2+\lambda_2 e^{-(\lambda_2+\mu_2)t}\right)$$

$$= \overset{\text{(regroup)}}{\cdots} = P_{0,0}^{(1)}(t)\overbrace{\left(-\lambda_2 e^{-(\lambda_2+\mu_2)t}\right)}^{\frac{d}{dt}P_{0,0}^{(2)}(t)}$$

(from expression for $P_{0,0}^{(2)}(t)$)

$$+ P_{0,0}^{(1)}(t)\left(-\frac{\lambda_1\mu_2}{\lambda_2+\mu_2}-\frac{\lambda_1\lambda_2}{\lambda_2+\mu_2}e^{-(\lambda_2+\mu_2)t}\right)$$

$$+ P_{0,0}^{(2)}(t)\,\lambda_1\, P_{1,0}^{(1)}(t)$$

$$= P_{0,0}^{(1)}(t)\left(\frac{d}{dt}P_{0,0}^{(2)}(t)\right)$$

$$+ P_{0,0}^{(2)}(t)\left(\frac{\lambda_1\mu_1}{\lambda_1+\mu_1}-\frac{\lambda_1\mu_1}{\lambda_1+\mu_1}e^{-(\lambda_1+\mu_1)t} - \frac{\lambda_1\mu_1}{\lambda_1+\mu_1}-\frac{\lambda_1\lambda_1}{\lambda_1+\mu_1}e^{-(\lambda_1+\mu_1)t}\right)$$

$$= P^{(1)}_{0,0}(t)\left(\frac{d}{dt}\, P^{(2)}_{0,0}(t)\right)$$

$$+ P^{(2)}_{0,0}(t)\, \underbrace{\left(-\lambda_1 e^{-(\lambda_1+\mu_1)t}\right)}_{= \frac{d}{dt} P^{(1)}_{0,0}(t)}$$

$$= P^{(1)}_{0,0}(t)\left(\frac{d}{dt}\, P^{(2)}_{0,0}(t)\right) + P^{(2)}_{0,0}(t)\left(\frac{d}{dt}\, P^{(1)}_{0,0}(t)\right)$$

$$= \frac{d}{dt}\left(P^{(1)}_{0,0}(t)\, P^{(2)}_{0,0}(t)\right) = \frac{d}{dt}\, P_{(0,0),(0,0)}(t)$$

product rule backwards

(★★

$= LHS.$

This proves that Kolmogorov's backward eqn. holds in this case.

The other state transitions & the forward equations are proved similarly.