Computing expectations:

Cont. case: $X \geq 0$, first moment

$$F(x) = \int_0^x \varphi(x)\,dx \;, \quad F'(x) = \varphi(x)$$

$$E[X] = \int_0^\infty x\,\varphi(x)\,dx$$

Integrate by parts: $u = x$, $v' = \varphi(x)$, $u' = 1$, $v = F(x) - 1$

$$= \lim_{c\to\infty} \int_0^c x\,\varphi(x)\,dx = \lim_{c\to\infty}\left(\left[x(F(x)-1)\right]_0^c - \int_0^c (F(x)-1)\,dx \right)$$

$$= \lim_{c\to\infty}\left(\underbrace{c\,(F(c)-1)}_{\to 0 \text{ first moment}} + \int_0^c \underbrace{[1-F(x)]}_{P[X>x]}\,dx \right) = \int_0^\infty P[X > x]\,dx$$

Discrete case: X taking values in $\{0, 1, 2, 3, \dots\}$

1 moment exists: $E[X] = \sum_{n=1}^{\infty} P[X \geq n]$.

Example (collecting coupons, football cards)

N football players: Prob of getting player i is p_i.

How long does it take to get a complete collection?

We buy cards according to a Poisson process of rate $\lambda \cdot N$.

N_i is the process you get by just counting cards with player i:

N_i is Poisson with rate $p_i\lambda$, and the N_i's are independent.

Let X_i be the first arrival time of N_i.

$$X = \max_i \{X_i\}$$

$$P[X \le t] = P[X_1 \le t \wedge X_2 \le t \wedge \dots \wedge X_n \le t]$$

$$\overset{\text{independence}}{=} P[X_1 \le t]\, P[X_2 \le t] \dots P[X_n \le t] \quad (\text{exponentially dist})$$

$$= (1-e^{-\lambda p_1 t})(1-e^{-\lambda p_2 t}) \dots (1-e^{-\lambda p_n t})$$

$$= \prod_{i=1}^{n} (1-e^{-\lambda p_i t})$$

Observe that $P[X > t] = 1 - \prod_{i=1}^{n}(1-e^{-\lambda p_i t})$

Hence:

$$E[X] = \int_0^\infty P[X>t]\,dt = \int_0^\infty \left(1-\prod_{i=1}^{n}(1-e^{-\lambda p_i t})\right) dt.$$

What about the number of cards we have to buy:

$\lambda = 1$: Let $T_1, T_2, \dots$ be the interarrival times of X.

$$X = T_1 + T_2 + \dots + T_{N} \leftarrow \text{when I have a complete collection}$$

$$= \sum_{i=1}^{\infty} \underbrace{\mathbb{1}_{\{i \le N\}}}_{\text{is 1 when } i \le N \text{ and 0 when } i > 0} T_i$$

Hence

$$E[X] = E\left[\sum_{i=1}^{\infty} \mathbb{1}_{\{i\le N\}} T_i\right] \overset{\text{independence}}{=} \sum_{i=1}^{\infty} \underbrace{E[\mathbb{1}_{\{i \le N\}}]}_{P[N \ge i]} \underbrace{E[T_i]}_{1}$$

$$= \sum_{i=1}^{\infty} P[N \ge i] = E[N]$$

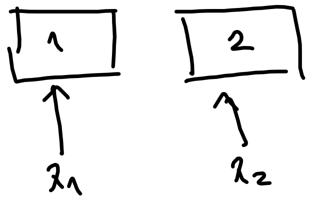
Thus

$$E[N] \overset{\lambda=1}{=} E[X] = \int_0^\infty \left(1-\prod_{i=1}^{n}(1-e^{-p_i t})\right) dt$$

Example: $N = 23$ $\quad p_i = \frac{1}{23}$ $\quad E[N] = 86$.

New problem: $S_n^{(1)}$ = queue nr n at 1

$S_m^{(2)}$ = —"— m at 2



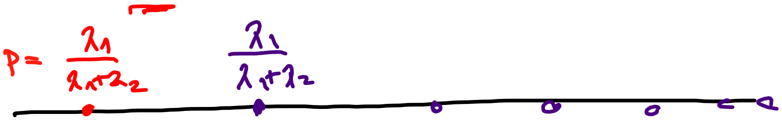
$P[S_n^{(1)} \le S_m^{(2)}]$

$n = m = 1$: $P[S_1^{(1)} \le S_1^{(2)}] = \dfrac{\lambda_1}{\lambda_1+\lambda_2}$

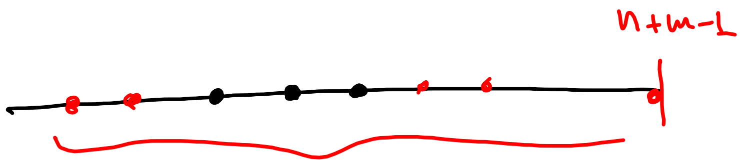
$n=2,\ m=1$ $\quad P[S_2^{(1)} \le S_1^{(2)}] = \dfrac{\lambda_1}{\lambda_1+\lambda_2} \cdot \dfrac{\lambda_1}{\lambda_1+\lambda_2} = \left(\dfrac{\lambda_1}{\lambda_1+\lambda_2}\right)^2$

$P[S_n^{(1)} \le S_m^{(2)}]$

$p = \dfrac{\lambda_1}{\lambda_1+\lambda_2}$



$q = \dfrac{\lambda_2}{\lambda_1+\lambda_2}$



$$\sum_{k=n}^{n+m-1} \binom{n+m-1}{k} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n+m-1-k}$$