9th March, 2023

STK2310

Mandatory assignment 1 of 1

Submission deadline

Thursday 30th March 2023, 14:30 in Canvas (canvas.uio.no).

Instructions

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course, and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. You only have one attempt at the assignment, and you need to have the assignments approved in order to take the exam. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) well before the deadline.

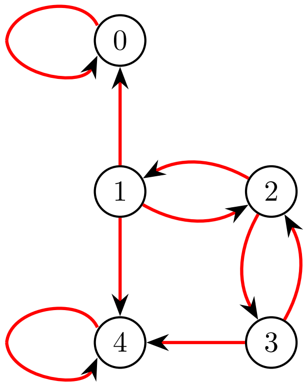
Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

Note: *From the fall semester of 2021, there is a new regime for mandatory assignments. In the new regime, you only have one attempt at each assignment and not two as in earlier years. As the purpose of the new regime is to handle the assignments in a more efficient and pedagogical way and not to fail more students, we are putting more emphasis on* effort *in the grading: As long as you have documented that you have made a serious attempt at the majority of the problems, we will pass you. The best way to document that you have tried, is, of course, to solve the problems, but you can also do it by telling us what you have tried and why it failed. We encourage you to discuss, collaborate, and help each other. Do not hesitate to contact the teachers (preferably well in advance of the deadline) if you have problems.*

You may use software freely to invert matrices, solve systems of equations etc.

Problem 1. A Markov chain has states $\{0, 1, 2, 3, 4\}$. The diagram shows the transitions between the states (there is an arrow from state i to state j if and only if $p_{ij} > 0$).



a) Find the communication classes of X and determine which of them are transient and which are recurrent.

b) Assume $p_{10} = \frac{1}{3}, p_{12} = \frac{1}{3}, p_{21} = \frac{1}{2}, p_{32} = \frac{1}{2}$ and find the transition matrix P of X. Use a computer program to compute P^n for large values of n and use the results to make a conjecture for the probability q_i that X started in a state i will eventually end up in the absorbing state 0.

c) Explain that the probabilities q_i in b) satisfies the following system of linear equations:

$$q_0 = 1 \tag{1}$$

$$q_1 = \frac{1}{3}q_0 + \frac{1}{3}q_2 + \frac{1}{3}q_4 \tag{2}$$

$$q_2 = \frac{1}{2}q_1 + \frac{1}{2}q_3 \tag{3}$$

$$q_3 = \frac{1}{2}q_2 + \frac{1}{2}q_4 \tag{4}$$

$$q_4 = 0 \tag{5}$$

Solve the system and prove your conjecture.

Problem 2. A Markov chain X has states $\{0, 1, 2, 3, 4\}$. The transition matrix is

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

a) Show that X is recurrent.

b) Show that X is time reversible and find the the stationary distribution $\pi = (\pi_0, \pi_1, \pi_2, \pi_3, \pi_4)$.

c) What is the expected number of times X started at 0 visits $i = 0, 1, 2, 3$, respectively, before it hits 4?

d) What is the expected number of steps X needs to get from 0 to 4?

e) What is the probability that X started at 0 hits 2 before it hits 4?

Problem 3. Think of $I = (0, 1]$ as the time interval from 0 to 1. Let n be a natural number and divide I into n subintervals $I_1 = (0, \frac{1}{n}]$, $I_2 = (\frac{1}{n}, \frac{2}{n}]$,..., $I_n = (\frac{n-1}{n}, 1]$. Assume that $\lambda < n$ and that the probability that there will occur an accident in the (time) interval $I_i = (\frac{i-1}{n}, \frac{i}{n}]$ is $\frac{\lambda}{n}$. Assume also that accidents in one interval are independent of accidents in all other intervals.

a) Show that the probability that accidents happen in exactly k intervals, $k \leq n$, is

$$P_k = \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}.$$

b) Show that

$$\lim_{n \to \infty} P_k = \frac{\lambda^k}{k!} e^{-\lambda}$$

and comment on the relationship with the Poisson distribution.

GOOD LUCK!