STK2130: Solution to the mandatory assignment 2023

Problem 1: a) As it is impossible to get from the states 0 and 4 to any other states, they are their own communication classes. For the remaining states 1, 2, and 3 we can always get from one state to any other state, and hence they form a communication class. This means that the communication classes are $\{0\}, \{1,2,3\}, \{4\}$. Of these, $\{0\}$ and $\{4\}$ are recurrent as the process can never leave these states, but $\{1,2,3\}$ is transient as the probability of returning to any of these states is less than 1 (there is a positive probability of getting trapped in 0 or 4).

b) As the total probability going out of a state is always 1, it is easy to compute the "missing" probabilities, and we get

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

I put the transition matrix into MATLAB and got:

$$P^{10} \approx \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.4257 & 0.0050 & 0 & 0.0050 & 0.5643 \\ 0.2821 & 0 & 0.0126 & 0 & 0.7053 \\ 0.1386 & 0.0075 & 0 & 0.0075 & 0.8464 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

I also tried

$$P^{100} \approx \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.4286 & 0.0000 & 0 & 0.0000 & 0.5714 \\ 0.2857 & 0 & 0.0000 & 0 & 0.7143 \\ 0.1429 & 0.0000 & 0 & 0.0000 & 0.8571 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

This makes it natural to guess that

$$\lim_{n\to\infty} P^n \approx \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.4286 & 0 & 0 & 0 & 0.5714 \\ 0.2857 & 0 & 0 & 0 & 0.7143 \\ 0.1429 & 0 & 0 & 0 & 0.8571 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The first column gives us the probabilities of eventually ending up in state 0 when starting from states 0, 1, 2, 3, 4, respectively, and hence our conjecture is: $q_0 = 1, q_1 \approx 0.4286, q_2 \approx 0.2857, q_3 \approx 0.1429, q_4 = 0$.

c) To explain the equations

$$q_0 = 1 \tag{1}$$
$$q_1 = \frac{1}{3}q_0 + \frac{1}{3}q_2 + \frac{1}{3}q_4 \tag{2}$$
$$q_2 = \frac{1}{2}q_1 + \frac{1}{2}q_3 \tag{3}$$
$$q_3 = \frac{1}{2}q_2 + \frac{1}{2}q_4 \tag{4}$$
$$q_4 = 0 \tag{5}$$

note that the first and the last one are immediate consequences of 0 and 4 being absorbing states. To explain the second equation, note that if we are in state 1, we will next go to each of the states 0, 2, and 4 with probability $\frac{1}{3}$. Hence the probability of eventually ending up in 0 starting at 1, must be the sum of $\frac{1}{3}$ times the probabilities of ending up in 0 starting at 0, 2, and 4, respectively, i.e.

$$q_1 = \frac{1}{3}q_0 + \frac{1}{3}q_2 + \frac{1}{3}q_4.$$

The third and fourth equations are obtained similarly by considering processes starting at states 2 and 3, respectively.

To solve the system, note that if we substitute the expressions for q_0 and q_4 into the second and fourth equation, we are left with three equations with three unknowns:

$$q_1 = \frac{1}{3} + \frac{1}{3}q_2 \tag{6}$$
$$q_2 = \frac{1}{2}q_1 + \frac{1}{2}q_3 \tag{7}$$
$$q_3 = \frac{1}{2}q_2 \tag{8}$$

This system is easily solved by substituting the first and third expression into the second, leaving us with

$$q_2 = \frac{1}{2} \cdot \left(\frac{1}{3} + \frac{1}{3}q_2\right) + \frac{1}{2} \cdot \frac{1}{2}q_2.$$

Solving for q_2, we get $q_2 = \frac{2}{7}$, which yields $q_3 = \frac{1}{7}$ and $q_1 = \frac{3}{7}$. As $\frac{3}{7} \approx 0.4286$, $\frac{2}{7} \approx 0.2857$, and $\frac{1}{7} \approx 0.1429$, this confirms our conjecture.

Problem 2: a) As all states communicate, the chain is irreducible, and since in addition the state space is finite, the Markov chain is recurrent by remark 2 on page 217.

b) The time reversibility conditions ("detailed balance equations") $\pi_{ij}p_i = \pi_j p_{ji}$ are in this case:

$$\pi_0 = \frac{1}{3}\pi_1, \;\; \frac{1}{3}\pi_1 = \frac{1}{2}\pi_2, \;\; \frac{1}{3}\pi_1 = \frac{1}{3}\pi_3, \;\; \frac{1}{2}\pi_2 = \frac{1}{3}\pi_3, \;\; \frac{1}{3}\pi_3 = \pi_4$$

Working from left to right, we may express all the π_i's in terms of π_0:

$$\pi_1 = 3\pi_0, \;\; \pi_2 = 2\pi_0, \;\; \pi_3 = 3\pi_0, \;\; \pi_4 = \pi_0$$

and it's easy to check that these values satisfy all the equations. We also need the π_i's to sum to 1:

$$1 = \pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 = \pi_0 + 3\pi_0 + 2\pi_0 + 3\pi_0 + \pi_0 = 10\pi_0$$

Hence $\pi_0 = \frac{1}{10}, \pi_1 = \frac{3}{10}, \pi_2 = \frac{2}{10}, \pi_3 = \frac{3}{10}, \pi_4 = \frac{1}{10}$. As the detailed balance equations are satisfied, the Markov chain is reversible.

The solutions to c)-e) are based on section 4.6 in the textbook, and I use the same notation as there.

c) As the three final questions are about what happens before we reach state 4, it's convenient to change the process slightly by making 4 an absorbing state. Hence we change the transition matrix into

$$\tilde{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Note that the corresponding Markov chain $\tilde{X}$ behaves exactly like X until it hits 4. For the new Markov chain, the state 4 is recurrent while the others are transient. The reduced transition matrix is therefore:

$$\tilde{P}_T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}$$

Using Matlab (or some other program), we compute

$$S = (I - \tilde{P}_T)^{-1} = \begin{pmatrix} \frac{8}{3} & 5 & \frac{8}{3} & 3 \\ \frac{5}{3} & 5 & \frac{8}{3} & 3 \\ \frac{4}{3} & 4 & \frac{10}{3} & 3 \\ 1 & 3 & 2 & 3 \end{pmatrix}$$

According to section 4.6, the component s_{ij} is the expected time $\tilde{X}$ spends in j when it is started in i. As we are starting in $i = 0$, the numbers we are interested in, are in the first row of the matrix: X spends an average time of $\frac{8}{3}$ in 0, 5 in 1, $\frac{8}{3}$ in 2, and 3 in 3 before reaching 4.

d) The average time X uses to reach 4 for the first time equals the sum of the average times spent in all the other states before reaching 4, i.e.: $\frac{8}{3} + 5 + \frac{8}{3} + 3 = \frac{40}{3}$.

e) According to the equation on the bottom line of page 246, the probability is:

$$f_{02} = \frac{s_{02}}{s_{22}} = \frac{\frac{8}{3}}{\frac{10}{3}} = \frac{4}{5}.$$

If you don't remember this formula, there is a more pedestrian way to solve the problem: Let q_i be the probability of hitting 2 before 4 if the process starts from state i. Then

$$\begin{aligned}
q_0 &= q_1 \\
q_1 &= \frac{1}{3}q_0 + \frac{1}{3}q_2 + \frac{1}{3}q_3 \\
q_2 &= 1 \\
q_3 &= \frac{1}{3}q_1 + \frac{1}{3}q_2 + \frac{1}{3}q_4 \\
q_4 &= 0
\end{aligned}$$

Solving this system of equations, we get $q_0 = \frac{4}{5}$.

Problem 3: a) There are $\binom{n}{k}$ ways to choose k intervals from n, and each such selection happens with probability $p^k(1-p)^{n-k}$. Hence

$$P_k = \binom{n}{k}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k}.$$

b) Note that

$$\begin{aligned}
P_k &= \binom{n}{k}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k} \\
&= \frac{n(n-1)\cdots(n-k+1)}{k!}\left(\frac{\lambda}{n}\right)^k\frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^k} \\
&= \frac{\lambda^k}{k!}\cdot 1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdot\ldots\cdot\left(1-\frac{k-1}{n}\right)\frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^k}
\end{aligned}$$

In this expression, both the finite products (remember that k is fixed)

$$1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdot\ldots\cdot\left(1-\frac{k-1}{n}\right)$$

and

$$\left(1-\frac{\lambda}{n}\right)^k$$

go to 1 as n goes to infinity, while

$$\left(1-\frac{\lambda}{n}\right)^n \to e^{-\lambda}.$$

Hence

$$\lim_{n\to\infty} P_k = \frac{\lambda^k}{k!}e^{-\lambda}$$

The result strengthens the intuition that the Poisson distribution with rate λ is a good model for counting events that happen with a frequency that (for short intervals) is λ times the length of the interval.