Continuous time Markov chains (ch4)

Setting: State space S: finite or countable, typically $S=\{0,1,2,\dots,N\}$, $S=\{0,1,2,\dots\}$

Time line: $[0,\infty)$

Process: $\{X_t\}_{t\in[0,\infty)}$, X_t r.v. with values in S.

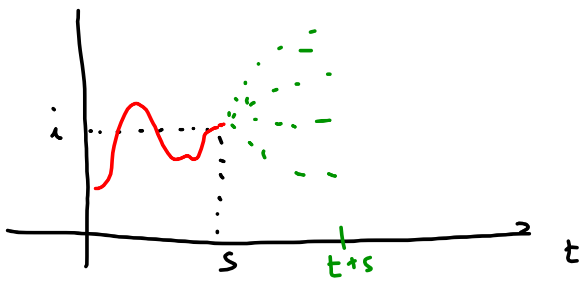
Let:

$$P_t(i,j) = P[X_t=j \mid X_0=i] \overset{\text{Notation}}{=} P_i[X_t=j]$$

Definition: X is a continuous time MC if

$$P_t(i,j) = P[X_{s+t}=j \mid X_s=i \wedge X_{s_1}=i_1, X_{s_2}=i_2,\dots,X_{s_n}=i_n]$$

for all $i,j,i_1,i_2,\dots,i_n$ and all $s_1,s_2,\dots,s_n<s$.



Chapman-Kolmogorov equations

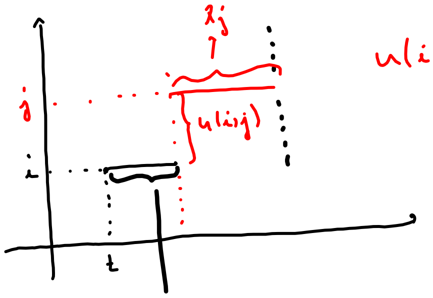
$$P_{t+s}(i,j) = \sum_{k\in S} P_s(i,k)\,P_t(k,j)$$

Proof: $P_{t+s}(i,j) = P_i[X(t+s)=j]$

$$= \sum_{k\in S} P_i[X_s=k]\,P_i[X(t+s)=j \mid X_s=k]$$

$$= \sum_{k\in S} P_s(i,k)\,P_t(k,j)$$

Alternative (dynamic) description



$u(i,i) = 0$

$\sum_{j \neq i} u(i,j) = 1$

exponential with λ_i

Two descriptions

I: $P_t(i,j)$

II: λ_i, $u(i,j)$

How to get from one to the other

Definition: For $i \neq j$, define

$$q_{ij} = \lim_{h \to 0} \frac{P_h(i,j)}{h} = \lim \frac{P_h(i,j) - P_0(i,j)}{h} \quad \text{(assume existence)}$$

Want to express q_{ij} in terms of λ_i and $u(i,j)$:

h small:

$$P_h(i,j) = P_i[X_h = j] = P[\underbrace{T_i}_{\text{exp rate } \lambda_i} < h]\, u(i,j) + o(h)$$

$$= (1 - e^{-\lambda_i h})\, u(i,j) + o(h)$$

$$= (\cancel{1} - (\cancel{1} - \lambda_i h + o(h)))\, u(i,j) + o(h)$$

$$= \lambda_i h\, u(i,j) + o(h)$$

Hence

$$q_{ij} = \lim_{h \to 0} \frac{P_h(i,j)}{h} = \lim_{h \to 0} \frac{\lambda_i h\, u(i,j) + o(h)}{h} = \lambda_i u(i,j) + 0$$

This means that $q_{ij} = \lambda_i u(i,j)$!

Note that: $\sum_{j \neq i} q_{ij} = \sum_{j \neq i} \lambda_i u(i,j) = \lambda_i \overbrace{\sum_{j \neq i} u(i,j)}^{1} = \lambda_i$

$$u(i,j) = \frac{q_{ij}}{\lambda_i}$$

Summing up:

(i) Know $P_t(i,j)$. Can compute $q_{ij} = \lim_{h \to 0} \frac{P_h(i,j)}{h}$ (for $i \neq j$).

(ii) Know q_{ij}, can compute $\lambda_i = \sum_{j \neq i} q_{ij}$, $u(i,j) = \frac{q_{ij}}{\lambda_i}$

(iii) know λ_i, u_{ij}, can compute $q_{ij} = \lambda_i u(i,j)$

(iv) Know q_{ij}, can we compute $P_t(i,j)$???

Kolmogorov backward/forward equation.

Derivation of the Kolmogorov backward equation $i \neq j$

$$\frac{P_{t+h}(i,j) - P_t(i,j)}{h} = \frac{1}{h}\left[\sum_k P_h(i,k)P_t(k,j) - P_t(i,j)\right]$$

$$= \frac{1}{h}\left[\sum_{k\neq i} P_h(i,k)P_t(k,j) + P_h(i,i)P_t(i,j) - P_t(i,j)\right]$$

$$= \sum_{k\neq i}\frac{P_h(i,k)}{h}P_t(k,j) + \frac{1}{h}\left[P_h(i,i)-1\right]P_t(i,j)$$

$$= \sum_{k\neq i}\frac{P_h(i,k)}{h}P_t(k,j) + \frac{1}{h}\left[-\sum_{k\neq i}P_h(i,k)\right]P_t(i,j)$$

$$= \sum_{k\neq i}\frac{P_h(i,k)}{h}P_t(k,j) - \sum_{k\neq i}\frac{P_h(i,k)}{h}P_t(i,j)$$

$$\sum_{k\neq i} q_{ik}P_t(k,j) - \underbrace{\left(\sum_{k\neq i} q_{ik}\right)}_{\lambda_i} P_t(i,j)$$

Hence

$$P_t'(i,j) = \sum_{k\neq i} q_{ik}P_t(k,j) - \lambda_i P_t(i,j) \leftarrow \text{Kolmogorov backward equations}$$

For each j, the function

$P(0,j), P(1,j), \ldots$ go into this system of equations

Matrix form:

$$P_t = \begin{bmatrix} P_t(0,0) & P_t(0,1) & \cdots \\ P_t(1,0) & P_t(1,1) & \\ \vdots & & \ddots \end{bmatrix}$$

$$P_t' = \begin{bmatrix} P_t'(0,0) & P_t'(0,1) & \cdots \\ P_t'(1,0) & \cdots & \end{bmatrix}$$

Infinitesimal generator:

$$R = Q = \begin{bmatrix} -\lambda_0 & q_{01} & q_{02} & \cdots \\ q_{10} & -\lambda_1 & q_{12} & \cdots \\ \vdots & & \ddots & \\ q_{n0} & q_{n1} & \cdots & -\lambda_n \end{bmatrix} \begin{matrix} \text{— sum } 0 \\ \text{— sum } 0 \\ \\ \text{— sum } 0 \end{matrix}$$

R ← in old exam problems

Kolmogorov backward equation on matrix form:

$$\boxed{P_t' = QP_t}$$

Example: States 1 rate λ $u(1,2)=1$ $q(i,j)=\lambda_i u(i,j)$

2 rate μ $u(2,1)=1$

$q(1,2)=\lambda\, u(1,2)=\lambda$

$q(2,1)=\mu\, u(2,1)=\mu$

$$Q=\begin{pmatrix}-\lambda & \lambda\\ \mu & -\mu\end{pmatrix}$$

Kolmogorov: $P_t'=Q\,P_t$

$$\begin{pmatrix}p_{11}'(t) & p_{12}'(t)\\ p_{21}'(t) & p_{22}'(t)\end{pmatrix}=\begin{pmatrix}-\lambda & \lambda\\ \mu & -\mu\end{pmatrix}\begin{pmatrix}p_{11}(t) & p_{12}(t)\\ p_{21}(t) & p_{22}(t)\end{pmatrix}$$

Multiplying out:

(1) $p_{11}'(t)=-\lambda p_{11}(t)+\lambda p_{21}(t)=-\lambda\left(p_{11}(t)-p_{21}(t)\right)$

(2) $p_{12}'(t)=-\lambda p_{12}(t)+\lambda p_{22}(t)=-\lambda\left(p_{12}(t)-p_{22}(t)\right)$

(3) $p_{21}'(t)=\mu p_{11}(t)-\mu p_{21}(t)=\mu\left(p_{11}(t)-p_{21}(t)\right)$

(4) $p_{22}'(t)=\mu p_{12}(t)-\mu p_{22}(t)=\mu\left(p_{12}(t)-p_{22}(t)\right)$

Try to solve (1)+(3): Subtract (1)-(3):

$$p_{11}'(t)-p_{21}'(t)=-(\lambda+\mu)\left(p_{11}(t)-p_{21}(t)\right)$$

Put $y(t)=p_{11}(t)-p_{21}(t)$, then

$$y'=-(\lambda+\mu)y \Rightarrow y=Ce^{-(\lambda+\mu)t}$$

$$p_{11}(t)-p_{21}(t)=Ce^{-(\lambda+\mu)t}$$

With $t=0$:

$$1=\underbrace{p_{11}(0)}_{1}-\underbrace{p_{21}(0)}_{0}=Ce^{-(\lambda+\mu)0}=C$$

Which means

$$p_{11}(t)-p_{21}(t)=e^{-(\lambda+\mu)t}\Rightarrow p_{21}(t)=p_{11}(t)-e^{-(\lambda+\mu)t}$$

Back to equation (1)

$$p_{11}'(t)=-\lambda\left(p_{11}(t)-p_{21}(t)\right)=-\lambda e^{-(\lambda+\mu)t}$$

Integrate

$$p_{11}(t)=\int-\lambda e^{-(\lambda+\mu)t}\,dt=\frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)t}+C$$

What is C:

$$1=p_{11}(0)=\frac{\lambda}{\lambda+\mu}\underbrace{e^{0}}_{1}+C\Rightarrow C=1-\frac{\lambda}{\lambda+\mu}=\frac{\lambda+\mu}{\lambda+\mu}-\frac{\lambda}{\lambda+\mu}=\frac{\mu}{\lambda+\mu}.$$

Hence

$$p_{11}(t)=\frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)t}+\frac{\mu}{\lambda+\mu}\longrightarrow\frac{\mu}{\lambda+\mu}$$

$$p_{21}(t)=-\frac{\mu}{\lambda+\mu}e^{-(\lambda+\mu)t}+\frac{\mu}{\lambda+\mu}\longrightarrow\frac{\mu}{\lambda+\mu}$$