Backward Kolmogorov:

$$p_t' = Q p_t$$

i.e.

$$p_t'(i,j) = \sum_{k \neq i} q_{ik} p_{kj}(t) - \lambda_i p_t(i,j)$$

i.e. one system for each.

Forward Kolmogorov:

Chapman-Kolmogorov

$$p_t'(i,j) = \lim_{h\to 0} \frac{p_{t+h}(i,j) - p_t(i,j)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\sum_k p_t(i,k) p_h(k,j) - p_t(i,j)\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\sum_{k\neq j} p_t(i,k) p_h(k,j) + p_t(i,j) p_h(j,j) - p_t(i,j)\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\sum_{k\neq j} p_t(i,k) p_h(k,j) + p_t(i,j)\left[p_h(j,j) - 1\right]\right]$$

$\sum p_h(j,k)$

$\sum_k p_h(j,k) = 1$

$$= \lim_{h\to 0} \frac{1}{h}\left[\sum_{k\neq j} p_t(i,k) p_h(k,j) - p_t(i,j)\sum_{k\neq j} p_h(j,k)\right]$$

$$= \lim_{h\to 0} \left[\sum_{k\neq j} p_t(i,k) \frac{p_h(k,j)}{h} - p_t(i,j)\sum_{k\neq j} \frac{p_h(j,k)}{h}\right]$$

$$= \sum_{k\neq j} p_t(i,k)\, q(k,j) - p_t(i,j) \overbrace{\sum_{k\neq j} q(j,k)}^{\lambda_j}$$

$$= \sum_{k\neq j} p_t(i,k)\, q(k,j) - \lambda_j p_t(i,j)$$

Kolmogorov's forward equations:

$$p_t'(i,j) = \sum_{k\neq j} p_t(i,k)\, q(k,j) - \lambda_j p_t(i,j)$$

Matrix form:

$$P_t' = P_t Q.$$

Observation: $P_t' = Q P_t$ backward

$P_t' = P_t Q$ forward

Hence $Q P_t = P_t Q$, P_t and Q commute!

Why? Assume P_t and Q take numbers as values, we have

$$P_t = e^{tQ}$$

The same is true in the general case if we define

$$e^{tQ} = \sum_{n=0}^{\infty} \frac{(tQ)^n}{n!}$$

Interlude integral:

$$I = \int \beta j\, e^{\beta j t} \left(1 - e^{-\beta t}\right)^{j-1} dt$$

$$= \int \beta j\, \underbrace{e^{\beta(j-1)t}}_{(e^{\beta t})^{j-1}} \cdot e^{\beta t} \left(1 - e^{-\beta t}\right)^{j-1} dt$$

$$u = e^{\beta t} - 1$$

$$du = \beta e^{\beta t} dt$$

$$= \int \beta j\, e^{\beta t} \left(e^{\beta t} - 1\right)^{j-1} dt$$

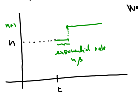
$$= \int j\, u^{j-1}\, du = u^{j} + C = \left(e^{\beta t} - 1\right)^{j} + C$$

$$\int \beta j\, e^{\beta j t} \left(1 - e^{-\beta t}\right)^{j-1} dt = \left(e^{\beta t} - 1\right)^{j} + C$$

Application to Yule processes

X_t models the size of a population of females at time t.

All individuals have the same birth rate β.



Want to find $p_t(i,j)$

Know $p_t(i,j) = 0$ if $j < i$.

$p_t(i,j)$ for $j \geq i$.

Rates: $\lambda_n = n\beta$

$u(i, i+1) = 1$

$u(i,j) = 0$ for $j \neq i+1$.

$q(i,i+1) = \lambda_i u(i,i+1)$

$= i\beta$

$q(i,j) = 0$ otherwise.

Kolmogorov forward equation: $\neq 0$ for $k = j-1$

$$p_t'(i,j) = \sum_{k \neq j} p_t(i,k) q_{kj} - \lambda_j p_t(i,j)$$

In this case

$$p_t'(i,j) = p_t(i,j-1)\, q_{j-1,j} - \lambda_j p_t(i,j)$$
$$= (j-1)\beta\, p_t(i,j-1) - j\beta p_t(i,j)$$

i.e

$$p_t'(i,j) = -j\beta p_t(i,j) + (j-1)\beta\, p_t(i,j-1)$$

Idea: Start with $j = 1$, and take the step upwards.

Concentrating on $p_t(1,j)$:

$$p_t'(1,1) = -\beta p_t(1,1) + 0 \qquad y = p_t(1,1)$$

$$y' = -\beta y \Rightarrow y = Ce^{-\beta t}$$

Finding C:

$$1 = p_0(1,1) = C \underbrace{e^{-\beta \cdot 0}}_{1} \Rightarrow C = 1$$

Hence $p_t(1,1) = e^{-\beta t}$

Next step is to find $p_t(1,2)$.

$e^{-\beta t}$

$$p_t'(1,2) = -2\beta p_t(1,2) + \beta p_t(1,1)$$

$$p_t'(1,2) = -2\beta p_t(1,2) + \beta e^{-\beta t} \qquad y = p_t(1,2)$$

$$y' + 2\beta y = \beta e^{-\beta t} \quad \Big|\, e^{2\beta t}$$

$$\underbrace{y' e^{2\beta t} + y \cdot 2\beta \cdot e^{2\beta t}}_{(ye^{2\beta t})' = \beta e^{\beta t}} = \beta e^{\beta t}$$

$\Downarrow$ integrate

$$ye^{2\beta t} = e^{\beta t} + C \quad \Big|\, e^{-2\beta t}$$

$$y = e^{-\beta t} + Ce^{-2\beta t} \qquad t = 0,\ p_0(1,2) = 0 \text{ forces } C = -1$$

Hence $p_t(1,2) = e^{-\beta t} + Ce^{-2\beta t}$

$$= e^{-\beta t} - e^{-2\beta t} = e^{-\beta t}(1 - e^{-\beta t})$$

Try $p_t(1,3)$:

$$p_t'(1,3) = -3\beta p_t(1,3) + 2\beta p_t(1,2) \qquad y = p_t(1,3)$$

i.e. $p_t'(1,3) + 3\beta p_t(1,3) = 2\beta e^{-\beta t}(1 - e^{-\beta t})$

$$y' + 3\beta y = 2\beta e^{-\beta t}(1 - e^{-\beta t}) \quad \Big|\, e^{3\beta t}$$

$$\underbrace{y' e^{3\beta t} + y 3\beta e^{3\beta t}}_{} = 2\beta e^{-2\beta t}(1 - e^{-\beta t})$$

$$(ye^{3\beta t})' = 2\beta e^{-2\beta t}(1 - e^{-\beta t}).$$

Integrate

$$ye^{3\beta t} = (e^{\beta t} - 1)^2 + C \quad \Big|\, e^{-3\beta t}$$

$$p_t(1,3) = e^{-3\beta t}(e^{\beta t} - 1)^2 + Ce^{-3\beta t}$$

$\boxed{t = 0,\ p_0(1,3) = 0 \Downarrow C = 0}$

$$= e^{-3\beta t}(e^{\beta t} - 1)^2$$
$$= e^{-\beta t} \cdot e^{-2\beta t}(e^{\beta t} - 1)^2$$
$$= e^{-\beta t}(1 - e^{-\beta t})^2$$

Hence $p_t(1,1) = e^{-\beta t}$, $p_t(1,2) = e^{-\beta t} \cdot (1 - e^{-\beta t})$

$p_t(1,3) = e^{-\beta t}(1 - e^{-\beta t})^2$

Guess: $p_t(1,j) = e^{-\beta t}(1 - e^{-\beta t})^{j-1}$

Proof by induction:

Assume $p_t(1,j) = e^{-\beta t}(1 - e^{-\beta t})^{j-1}$

Prove: $p_t(1,j+1) = e^{-\beta t}(1 - e^{-\beta t})^{j}$

Forward Kolmogorov:

$$p_t'(1,j+1) = -\beta(j+1) p_t(1,j+1) + \beta j\, p_t(1,j) \qquad y = p_t(1,j+1)$$

$$y' + \beta(j+1) y = \beta j e^{-\beta t}(1 - e^{-\beta t})^{j-1} \quad \Big|\, e^{\beta(j+1)t}$$

$$\underbrace{y' e^{\beta(j+1)t} + y \beta(j+1) e^{\beta(j+1)t}}_{} = \beta j e^{\beta j t}(1 - e^{-\beta t})^{j-1}$$

$$(y e^{\beta(j+1)t})' = \beta j e^{\beta j t}(1 - e^{-\beta t})^{j-1}$$

Interlude integral!

Integrate

$$y e^{\beta(j+1)t} = (e^{\beta t} - 1)^j + C$$

$$y = \underbrace{e^{-\beta(j+1)t}(e^{\beta t} - 1)^j}_{} + Ce^{-\beta(j+1)t} \qquad C = 0$$

$$p_t(1,j) = e^{-\beta t}(e^{-\beta t})^j (e^{\beta t} - 1)^j$$
$$= e^{-\beta t}(1 - e^{-\beta t})^j$$

Hence: $p_t(1,j) = e^{-\beta t}(1 - e^{-\beta t})^j$

What about $p_t(i,j)$, i.e starting with i individuals

Start i process at 1:

$X_1 + X_2 + \dots + X_i \leftarrow$ independent process starting at 1

$$P_i[X_t = j] = P[\underbrace{X_1^{(t)} + \dots + X_i^{(t)}}_{} = j] \qquad i \to j$$

$n_1, n_2, \dots, n_i \qquad n_1 + n_2 + \dots + n_i = j \qquad n_k \geq 1$

$X_1(t) = n_1,\ X_2(t) = n_2, \dots, X_i(t) = n_i$

$$\prod_{k=1}^{i} \underbrace{e^{-\beta t}(1 - e^{-\beta t})^{n_k}}_{} = e^{-i\beta t}(1 - e^{-\beta t})^{j-i}$$

In how many ways can we find number $n_1, n_2, \dots, n_i$ such that $n_1 + n_2 + \dots + n_i = j$ and $n_k \geq 1$?

$i-1$

$$\underbrace{\cdot\ \cdot \mid \cdot\ \cdot\ \cdot \mid \cdot \mid \cdot\ \cdot\ \cdot\ \cdot \mid \cdot\ \cdot}_{j}$$

can put $i-1$ division markers into $j-1$ spots. $\binom{j-1}{i-1}$

$$p_t(i,j) = \binom{j-1}{i-1} e^{-\beta i t}(1 - e^{-\beta t})^{j-i}$$

Section: 4.3 Limit behavior