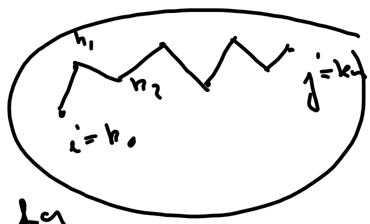
Stationary distributions

Definition: A cont. time Markov chain is *irreducible* if any time we pick two states i, j we can find a sequence $k_0, k_1, k_2, \ldots, k_n$ such that $k_0 = i$, $k_n = j$ and

$$q(k_{\ell-1}, k_\ell) > 0$$



Proposition: If X is irreducible then for all states i, j, the probability

$$p_t(i,j) > 0 \quad \text{for } t > 0.$$

Proof: Let $k_0, k_1, \ldots, k_n$ be a path of the type above from i to j such that $k_{\ell-1} \neq k_\ell$ for ℓ. By definition

$$0 < q(k_{\ell-1}, k_\ell) = \lim_{h \to 0} \frac{p_h(k_{\ell-1}, k_\ell)}{h},$$

and hence $p_h(k_{\ell-1}, k_\ell) > 0$ for all sufficiently small h.

Hence there is an $\varepsilon > 0$ such that if $h \leq \varepsilon$, then

$$p_h(k_{\ell-1}, k_\ell) > 0 \text{ for all } \ell.$$

If $t \leq \varepsilon$, we then have

$$p_t(i,j) \geq p_{t/n}(i, k_1)\, p_{t/n}(k_1, k_2) \cdots p_{t/n}(k_{n-1}, k_n) > 0.$$

with $t/n \leq \varepsilon$ in each factor.

What with a large t: Choose N so large that $t/N < \varepsilon$. Then

$$p_t(i,j) \geq \underline{p_{t/N}(i,j)}\, \underline{p_{t/N}}(j,j) \cdots p_{t/N}(j,j) > \underline{\underline{0}}$$

Distribution: $\pi = (\pi_1, \pi_2, \pi_3, \dots)$ $0 \le \pi_i \le 1$ and $\sum \pi_i = 1$

Starting distribution: $\pi_i = P[X_0 = i]$

Distribution at time t:

$(\pi_t)_j = P[X_t = j \mid \pi \text{ starting distribution}] = \pi \cdot P_t$

Definition: π is stationary if $\pi_t = \pi$ at all $t \ge 0$.

Theorem: π is a stationary distribution if and only if $\pi Q = 0$.

Intuitive explanation:

$$(\pi Q)_j = \sum_i \pi_i Q_{ij} = \underbrace{\sum_{i \neq j} \pi_i q_{ij}}_{\text{rate of mass entering } j} - \underbrace{\lambda_j \pi_j}_{\text{rate of mass leaving } j} = 0 \text{ when stationary}$$

$\boxed{\text{Forward Kolmogorov } P_t' = P_t Q}$

Proof: Assume that π is stationary:

$$0 = (\pi_t)' = (\pi P_t)' = \pi P_t' = \pi P_t Q = \pi_t Q = \pi Q$$

Assume next that $\pi Q = 0$.

Backward Kolmogorov $P_t' = Q P_t$

$$(\pi_t)' = (\pi P_t)' = \pi P_t' = \underbrace{\pi Q}_{0} P_t = 0.$$

Hence π_t is a constant, $\pi_t = \pi_0 = \pi$, thus π is stationary.

Theorem: If X is irreducible and has a stationary distribution π, then for i, $\lim_{t\to\infty} P_t(i,j) = \pi_j$, i.e.

$$P_i[X(t)=j] \to \pi_j$$

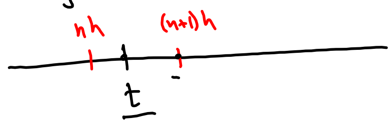
Proof: For any h, then the transition matrix $P_h = (P_h(i,j))$ defines a discrete time Markov chain which is irreducible, aperiodic and has π as a stationary distribution. Hence by the discrete time theory

$$P_{nh}(i,j) \to \pi_j \quad \text{when } n\to\infty$$

We need to prove that

$$P_t(i,j) \to \pi_j \quad \text{when } j\to\infty.$$

For any t and h, we have



Note that: $P_t(i,j) \ge P_{nh}(i,j)\cdot e^{-\lambda_j(t-nh)} \ge P_{nh}(i,j)e^{-\lambda_j h}$

$P_{(n+1)h}(i,j) \ge P_t(i,j)e^{-\lambda_j[(n+1)h-t]} \ge P_t(i,j)e^{-\lambda_j h}$

Hence

$$\underset{\downarrow\ \pi_j\ \ \downarrow\ 1}{P_{nh}(i,j)e^{-\lambda_j h}} \le \underset{\downarrow\ \pi_j}{P_t(i,j)} \le \underset{\downarrow\ \pi_j\ \ \downarrow\ 1}{P_{(n+1)h}(i,j)e^{\lambda_j h}}$$

Given an $\varepsilon>0$, I have to find T such that if $t\ge T$, then

$$\pi_j-\varepsilon < P_t(i,j) < \pi_j+\varepsilon.$$

First choose h such that

$$e^{-\lambda_j h} > 1-\frac{\varepsilon}{4}$$
$$e^{\lambda_j h} < 1+\frac{\varepsilon}{4}.$$

Next choose T such that if $nh > T-h$, then

$$\pi_j - \frac{\varepsilon}{4} < P_{nh}(i,j) < \pi_j+\frac{\varepsilon}{4}$$

The

$$P_t(i,j) \le P_{(n+1)h}(i,j)e^{\lambda_j h} < \left(\pi_j+\frac{\varepsilon}{4}\right)\left(1+\frac{\varepsilon}{4}\right)$$
$$= \pi_j + \underbrace{(1+\pi_j)}_{\le 2}\frac{\varepsilon}{4} + \frac{\varepsilon^2}{16} < \pi_j+\varepsilon$$

$$P_t(i,j) \ge P_{nh}(i,j)e^{-\lambda_j h} \ge \left(\pi_j-\frac{\varepsilon}{4}\right)\left(1-\frac{\varepsilon}{4}\right) > \pi_j-\varepsilon.$$

Hence $\lim_{t\to\infty} P_t(i,j) = \pi_j$ HOORAY!

Example: LA weather (parody)

1 Sunny - exponential, mean 3, $\longrightarrow$ smoggy

2 Smoggy - exponential, mean 4, $\longrightarrow$ rainy

3 Rainy - exponential, mean 1, $\rightarrow$ sunny

$q_{ij} = \lambda_i u(i,j)$

$$\left.\begin{array}{l} \underline{1} \quad \lambda_1 = \frac{1}{3}, \quad u(1,2) = 1 \\ \qquad\qquad\quad u(1,3) = \underline{0} \end{array}\right\} \quad \begin{array}{l} q(1,2) = \lambda_1 u(1,2) = \frac{1}{3} \\ q(1,3) = 0 \end{array}$$

$$\left.\begin{array}{l} \underline{2} \quad \lambda_2 = \underline{\frac{1}{4}}, \quad u(2,3) = 1 \\ \qquad\qquad\quad u(2,1) = \underline{0} \end{array}\right\} \quad \begin{array}{l} q(2,3) = \lambda_2 u(2,3) = \frac{1}{4} \\ q(2,0) = 0 \end{array}$$

$$\left.\begin{array}{l} \underline{3} \quad \lambda_3 = \frac{1}{1} = 1 \quad u(3,0) = 1 \\ \qquad\qquad\quad u(3,1) = 0 \end{array}\right\} \quad \begin{array}{l} q(3,1) = 1 \cdot u(3,0) = 1 \\ q(3,2) = 0 \end{array}$$

$$Q = \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & -\frac{1}{4} & \frac{1}{4} \\ 1 & 0 & -1 \end{bmatrix}$$

Stationary distribution: $\pi Q = 0$

$$0 = \pi Q = (\pi_1, \pi_2, \pi_3) \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & -\frac{1}{4} & \frac{1}{4} \\ 1 & 0 & -1 \end{pmatrix} = \left(-\frac{\pi_1}{3} + \pi_3, \frac{\pi_1}{3} - \frac{\pi_2}{4}, \frac{\pi_2}{4} - \pi_3\right)$$

i.e. $\underbrace{-\frac{\pi_1}{3} + \pi_3 = 0}_{\pi_1 = 3\pi_3}$, $\frac{\pi_1}{3} - \frac{\pi_2}{4} = 0$, $\underbrace{\frac{\pi_2}{4} - \pi_3 = 0}_{\pi_2 = 4\pi_3}$

Also:

$$1 = \pi_1 + \pi_2 + \pi_3 = 3\pi_3 + 4\pi_3 + \pi_3 = 8\pi_3 \Rightarrow \pi_3 = \frac{1}{8}$$

$$\pi_1 = \frac{3}{8}$$

$$\pi_2 = \frac{4}{8} = \frac{1}{2}$$

Stationary distribution: $\pi = \underline{\left(\frac{3}{8}, \frac{1}{2}, \frac{1}{8}\right)}$

$$P_t(i,j) \xrightarrow[t \to \infty]{} \pi_j$$

$P_t(i,1) \to \frac{3}{8}$ sun

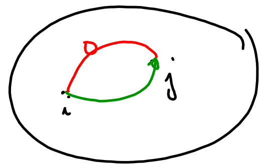
$P_t(i,2) \to \frac{1}{2}$ smog

$P_t(i,3) \to \frac{1}{8}$ rain

Detailed balance equation

The distribution π satisfies the detailed balance equation if for all i, j

$$\underbrace{\pi_i q_{ij}}_{\text{rate of mass going from } i \text{ to } j} = \underbrace{\pi_j q_{ji}}_{\text{rate of mass from } j \text{ to } i}$$



Proposition: If π satisfies the detailed balance equation, then π is a stationary distribution

Proof. Need to check that $\pi Q = 0$:

$$(\pi Q)_j = \sum_i \pi_i Q_{ij} = \sum_{i \neq j} \pi_i q_{ij} - \pi_j \lambda_j$$

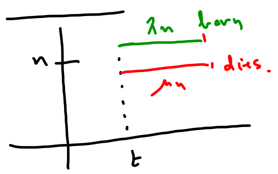
$$= \sum_{i \neq j} \pi_j q_{ji} - \pi_j \lambda_j = \pi_j \underbrace{\Big(\sum_{i \neq j} q_{ij}\Big)}_{=\lambda_j} - \pi_j \lambda_j$$

$$= \pi_j \lambda_j - \pi_j \lambda_j = 0.$$

Birth and death process

X_t – number of individuals in a pop. at time t.

$X_t = n$



First time something happens is exponential with rate $\lambda_n + \mu_n$

born $\frac{\lambda_n}{\mu_n + \lambda_n}$ $\qquad$ $\frac{\mu_n}{\mu_n + \lambda_n}$

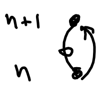
Markov: rate $\lambda_n + \mu_n$

$u(n, n+1) = \frac{\lambda_n}{\mu_n + \lambda_n}$ $\qquad q(n, n+1) = \lambda_n$

$u(n, n-1) = \frac{\mu_n}{\mu_n + \lambda_n}$ $\qquad q(n, n-1) = \mu_n.$

Detailed balance equation

$$\pi_i q_{ij} = \pi_j q_{ji}$$



$$\pi_{n+1} q_{n+1,n} = \pi_{n+1} \mu_{n+1}$$

$$\pi_n q_{n,n+1} = \pi_n \lambda_n$$

Detailed balance:

$$\pi_{n+1} \mu_{n+1} = \pi_n \lambda_n$$

$$\pi_{n+1} = \frac{\lambda_n}{\mu_{n+1}} \pi_n = \frac{\lambda_n \lambda_{n-1}}{\mu_{n+1} \mu_n} \pi_{n-1}$$

$$= \frac{\lambda_n \lambda_{n-1} \cdots \lambda_0}{\mu_{n+1} \mu_n \cdots \mu_1} \pi_0$$

Hence

$$\pi_n = \frac{\lambda_{n-1} \lambda_{n-2} \cdots \lambda_0}{\mu_n \mu_{n-1} \cdots \mu_1} \pi_0$$

Want

$$1 = \sum \pi_n = \Big(\underbrace{\sum \frac{\lambda_{n-1} \lambda_{n-2} \cdots \lambda_0}{\mu_n \mu_{n-1} \cdots \mu_1}}_{R} \Big) \pi_0 \Rightarrow \pi_0 = \frac{1}{R}$$

Result: $\sum_{n=1}^{\infty} \frac{\lambda_{n-1} \lambda_{n-2} \cdots \lambda_0}{\mu_n \mu_{n-1} \cdots \mu_1} < \infty$, then we have a stationary distribution satisfying the detailed balance equation.