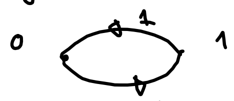
Convergence

Question: X has stat. dist π.

When will

$$p^{(n)}(i,j) \to \pi(j)\,?$$

Example:



$\pi(0) = \pi(1) = \frac{1}{2}$.

$$P^{(n)}_{00} = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

If $x \in S$, let

$$I_x = \{ n : p^{(n)}_{xx} > 0 \}$$

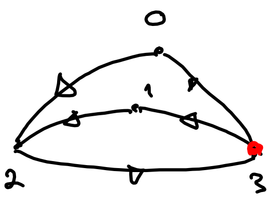
The period of x is the largest common divisor of elements in I_x

If the period 1, the chain is aperiodic

Theorem: Period is a class property.

Example: $S = \{0,1,2,3\}$

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix}$$



Class: $\{0,1,2,3\}$

Period of 3 is 3, and the same holds for all other states.

$$\pi = \left(\tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{3}, \tfrac{1}{3} \right)$$

$P^{(n)}_{33} = 0$ unless n is a multiple of 3

Theorem: If X is recurrent, irreducible, aperiodic and has a stationary distr. π, then

$$\lim_{n \to \infty} P^{(n)}_{ij} = \pi(j)$$

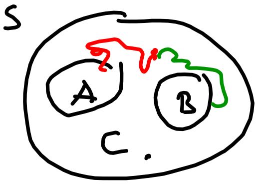
Ergodic Theorem: If X is irreducible with a stationary distribution π, and $\sum_{x \in S} |f(x)| \pi(x) < \infty$, then

$$\frac{1}{N} \sum_{n=0}^{N-1} f(X_n) \to \sum_{x \in S} f(x) \pi(x)$$

time average $\qquad$ space average

No assumption of periodicity.

Exit distribution

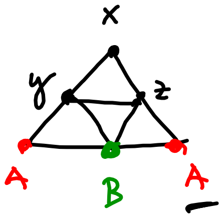


What is the probability of hitting A before B when we start in x? $h(x)$.

$$h(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in B \\ \sum_{y \in S} h(y)\, p_{xy}. \end{cases}$$

one equation for each $x \in C$, $|C|$ unknowns

Example:



Random walk on the graph – same probabilities to all neighbors.

Equation:

$$x = \tfrac{1}{2} y + \tfrac{1}{2} z$$

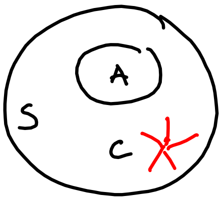
$$y = \tfrac{1}{4} x + \tfrac{1}{4} z + \tfrac{1}{4} \cdot 1 + \tfrac{1}{4} \cdot 0 = \tfrac{1}{4} x + \tfrac{1}{4} z + \tfrac{1}{4}$$

$$z = \tfrac{1}{4} x + \tfrac{1}{4} y + \tfrac{1}{4} \cdot 1 + \tfrac{1}{4} \cdot 0 = \tfrac{1}{4} x + \tfrac{1}{4} y + \tfrac{1}{4}$$

Three equations with 3 unknowns

$$x = \tfrac{1}{2},\ y = \tfrac{1}{2},\ z = \tfrac{1}{2}.$$

Exit times.



Starting at i, how long $\overbrace{\text{does it take}}^{g(i)}$ to exit C and hit A.

$$\underline{g(\underline{i})} = \begin{cases} 0 & \text{if } i \in A \\ \underline{1 + \sum_{j \in C} p(i,j)\, g(j)} & \end{cases} \quad \Bigg\} \begin{array}{l} |C| \text{ equations with} \\ |C| \text{ unknowns.} \end{array}$$

The reduced matrix r is the matrix we get from P by deleting all rows and colums that do not belong to C.

Matrix form as

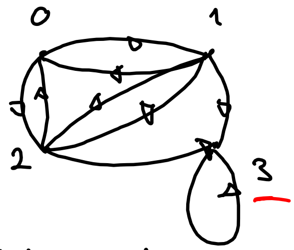
$$(I - r)g = \mathbb{1} \leftarrow \text{vector with } 1 \text{ in all components.}$$

Hence $g = (I-r)^{-1}\mathbb{1}$

Example:

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1/3 & 0 & 1/3 & 1/3 \\ 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

State diagram:



Comm. classes

$\{0,1,2\}$ transient

$\{3\}$ recurrent

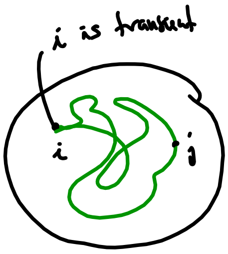
How long does it take to reach $\underline{3}$ in average.

$$r = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & \cancel{0} \\ \frac{1}{3} & 0 & \frac{1}{3} & \cancel{\frac{1}{3}} \\ \frac{1}{3} & \frac{1}{3} & 0 & \cancel{\frac{1}{3}} \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{1} \end{pmatrix}$$

$$(I-r)^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{3} & 1 & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & 1 \end{pmatrix}^{-1} = \underline{\begin{pmatrix} 2 & \frac{3}{2} & \frac{3}{2} \\ 1 & \frac{15}{8} & \frac{9}{8} \\ 1 & \frac{9}{8} & \frac{15}{8} \end{pmatrix}}$$

Hence

$$(I-r)^{-1}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 4 \end{pmatrix} \begin{array}{l} \leftarrow \text{number of steps starting } 0 \\ \leftarrow \text{——— " ——— } 1 \\ \leftarrow \text{——— " ——— } 2. \end{array}$$



Number of visits

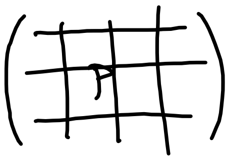
If j is transient, let s_{ij} be the number of visits to j when starting from i.

$$s_{ij} = \delta_{ij} + \sum_{k \in T} p_{ik} s_{kj}$$

↑ transient states

Reduced matrix

$P_T =$



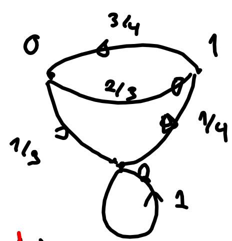
delete all "recurrent" rows and columns

$S = (s_{ij})$, i, j are transient.

$$S = I + P_T S$$

$$(I - P_T) S = I \Rightarrow S = (I - P_T)^{-1}$$

Example:



$$P = \begin{pmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 1 \end{pmatrix}$$

$$P_T = \begin{pmatrix} 0 & \frac{2}{3} & \cancel{\frac{1}{3}} \\ \frac{3}{4} & 0 & \cancel{\frac{1}{4}} \\ \cancel{0} & \cancel{0} & \cancel{1} \end{pmatrix} = \begin{pmatrix} 0 & \frac{2}{3} \\ \frac{3}{4} & 0 \end{pmatrix}$$

$$S = (I - P_T)^{-1} = \begin{pmatrix} 1 & -\frac{2}{3} \\ -\frac{3}{4} & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & \frac{4}{3} \\ \frac{3}{2} & 2 \end{pmatrix}$$

$s_{01} = \frac{4}{3}$ - the average number of visits to 1 if you start at 0.

Exponential random variables

Definition: A random variable T taking values in $[0,\infty)$ is called exponential with rate λ if

$$P[T \leq t] = 1 - e^{-\lambda t}$$

Density: $f(t) = \lambda e^{-\lambda t}$

Mean: $E[T] = \frac{1}{\lambda}$

Variance: $\mathrm{Var}(T) = \frac{1}{\lambda^2}$

Memoryless: $P[T > t+s \mid T > t] = P[T > s]$

Theorem: Assume that $T_1, T_2, \dots, T_n$ are independent and exponential distr. with rates $\lambda_1, \lambda_2, \dots, \lambda_n$. Then

$$V = \min\{T_1, T_2, \dots, T_n\}$$

is exponential with rate $\lambda = \lambda_1 + \lambda_2 + \dots + \lambda_n$. Also the probability that $V = T_i$ is

$$\frac{\lambda_i}{\lambda_1 + \lambda_2 + \dots + \lambda_n}$$

Example: Two taxi companies: mean 5 min company A, $\lambda_1 = \frac{1}{5}$; mean 3 min company B, $\lambda_2 = \frac{1}{3}$

How long wait for the first taxi (in average)?

Rate: $\lambda = \lambda_1 + \lambda_2 = \frac{1}{5} + \frac{1}{3} = \frac{3}{15} + \frac{5}{15} = \frac{8}{15}$.

Mean waiting time: $\frac{1}{\frac{8}{15}} = \frac{15}{8}$

What is the prob. that the first taxi to arrive is from company B?

$$\frac{\lambda_2}{\lambda_1 + \lambda_2} = \frac{\frac{1}{3}}{\frac{8}{15}} = \frac{1}{3} \cdot \frac{15}{8} = \frac{5}{8}$$

Theorem: Assume that $T_1, T_2, \dots, T_n$ are exponential, independent and have the same rate λ. Then

$$T = T_1 + T_2 + \dots + T_n$$

is gamma (λ, n)-distributed, i.e. it has density

$$f_T(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} \quad \text{for } t \geq 0.$$