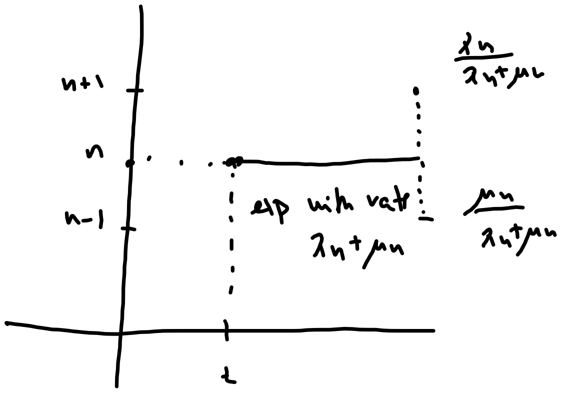
Birth and death processes



X_t - size of pop. at time t

if $X_t = n$ $\begin{cases} \lambda_n & \text{birth rate} \\ \mu_n & \text{death rate} \end{cases}$

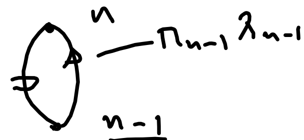
rate $\lambda_n + \mu_n$

$u(n, n+1) = \frac{\lambda_n}{\lambda_n+\mu_n}$, $q(n,n+1) = \lambda_n$

$u(n, n-1) = \frac{\mu_n}{\lambda_n+\mu_n}$, $q_n(n,n-1) = \mu_n$

Detailed balance distribution π:

$\pi_n \mu_n$



$$\pi_n \mu_n = \pi_{n-1} \lambda_{n-1}$$

$$\pi_n = \frac{\lambda_{n-1}}{\mu_n} \pi_{n-1} \quad (*)$$

$$\pi_0, \quad \pi_1 = \frac{\lambda_0}{\mu_1} \pi_0, \quad \pi_2 = \frac{\lambda_1}{\mu_2} \pi_1 = \frac{\lambda_1 \lambda_0}{\mu_2 \mu_1} \pi_0$$

etc $$\pi_n = \frac{\lambda_{n-1} \lambda_{n-2} \cdots \lambda_0}{\mu_n \mu_{n-1} \cdots \mu_1} \pi_0$$

Need in addition

$$1 = \sum_{n=0}^{\infty} \pi_n = \pi_0 + \frac{\lambda_0}{\mu_1} \pi_0 + \dots + \frac{\lambda_{n-1} \lambda_{n-2} \cdots \lambda_0}{\mu_n \mu_{n-1} \cdots \mu_1} \pi_0 + \cdots$$

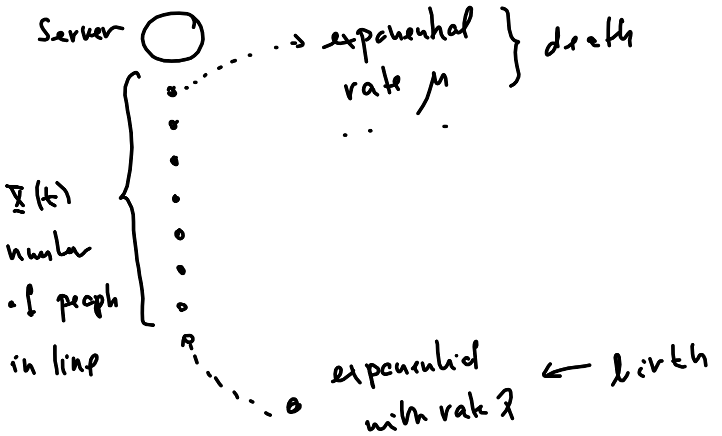
$$= \pi_0 \left(1 + \sum_{n=1}^{\infty} \frac{\lambda_{n-1} \lambda_{n-2} \cdots \lambda_0}{\mu_n \mu_{n-1} \cdots \mu_1} \right)$$

Given that $R = 1 + \sum_{n=1}^{\infty} \frac{\lambda_{n-1} \lambda_{n-2} \cdots \lambda_0}{\mu_n \mu_{n-1} \cdots \mu_1} < \infty$, then I need

$1 = \pi_0 R \Rightarrow \pi_0 = \frac{1}{R}$. Then

$$\pi_n = \frac{1}{R} \frac{\lambda_{n-1} \lambda_{n-2} \cdots \lambda_0}{\mu_n \mu_{n-1} \cdots \mu_1}.$$

Example (queuing theory):



$\lambda_n = \lambda$

$\mu_n = \mu$

$$R = 1 + \sum_{n=1}^{\infty} \frac{\lambda_{n-1} \cdots \lambda_0}{\mu_n \cdots \mu_1} = 1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{\mu^n} = \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n$$

$$\underset{\lambda < \mu}{=} \frac{1}{1 - \frac{\lambda}{\mu}} = \frac{\mu}{\mu - \lambda}.$$

Conclusion: If $\lambda < \mu$, then there is a stationary distribution

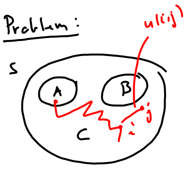
with $R = \frac{\mu}{\mu - \lambda}$

$$\Pi_0 = \frac{1}{R} = \frac{1}{\frac{\mu}{\mu-\lambda}} = \frac{\mu - \lambda}{\mu}.$$

$$\Pi_n = \frac{\lambda_{n-1} \cdots \lambda_0}{\mu_n \cdots \mu_1} \frac{1}{R} = \frac{\lambda^n}{\mu^n} \cdot \frac{1}{R} = \underline{\underline{\frac{\mu - \lambda}{\mu} \left(\frac{\lambda}{\mu}\right)^n}}$$

Exit distribution

Problem:



What is the probability of hitting A before B when starting at $i \in C$?

$h(i) = 1$ if $i \in A$

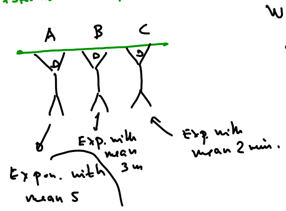
$h(i) = 0$ if $i \in B$

$h(i) = \sum_{j \in S} u(i,j)\, h(j)$ if $i \in C$

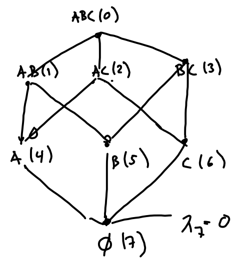
one equation for each $i \in C$ and one unknown $h(i)$ for each $i \in C$.

Example (Musketeers murder)

What is the prob of A winning.



Continuous time Markov chain:



0 ABC: $\lambda_0 = \lambda_C + \lambda_B + \lambda_A = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} = \frac{15+10+6}{30} = \frac{31}{30}$

$u(0,1) = \frac{\lambda_C}{\lambda_C+\lambda_B+\lambda_A} = \frac{\frac{1}{2}}{\frac{31}{30}} = \frac{15}{31}$, $q(0,1) = \lambda_0 u(0,1) = \frac{31}{30}\cdot\frac{15}{31} = \frac{1}{2}$

$u(0,2) = \frac{\lambda_B}{\lambda_C+\lambda_B+\lambda_A} = \frac{\frac{1}{3}}{\frac{31}{30}} = \frac{10}{31}$, $q(0,2) = \lambda_0 u(0,2) = \frac{31}{30}\cdot\frac{10}{31} = \frac{1}{3}$

$u(0,3) = \frac{\lambda_A}{\lambda_C+\lambda_B+\lambda_A} = \frac{\frac{1}{5}}{\frac{31}{30}} = \frac{6}{31}$, $q(0,3) = \lambda_0 u(0,3) = \frac{1}{5}$

1 AB: $\lambda_1 = \lambda_A + \lambda_B = \frac{1}{5} + \frac{1}{3} = \frac{3+5}{15} = \frac{8}{15}$

$u(1,4) = \frac{\lambda_B}{\lambda_A+\lambda_B} = \frac{\frac{1}{3}}{\frac{8}{15}} = \frac{5}{8}$, $q(1,4) = \lambda_1 u(1,4) = \frac{8}{15}\cdot\frac{5}{8} = \frac{1}{3}$

$u(1,5) = \frac{\lambda_A}{\lambda_A+\lambda_B} = \frac{3}{8}$, $q(1,5) = \frac{1}{5}$

2 AC: $\lambda_2 = \lambda_A + \lambda_C = \frac{1}{5} + \frac{1}{2} = \frac{2+5}{10} = \frac{7}{10}$

$u(2,4) = \frac{\lambda_C}{\lambda_A+\lambda_C} = \frac{5}{7}$, $q(2,4) = \frac{1}{2}$

$u(2,6) = \frac{\lambda_A}{\lambda_A+\lambda_C} = \frac{2}{7}$, $q(2,6) = \frac{1}{5}$

3 BC: $\lambda_3 = \lambda_B + \lambda_C = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$

$u(3,5) = \frac{3}{5}$ $q(3,5) = \frac{1}{2}$

$u(3,6) = \frac{2}{5}$ $q(3,6) = \frac{1}{3}$

Equations:

$h(0) = h(1)\frac{15}{31} + h(2)\frac{10}{31} + h(3)\frac{6}{31}$

$h(1) = \underbrace{h(4)}_{=1}\cdot\frac{5}{8} + \underbrace{h(5)}_{=0}\frac{3}{8} = \frac{5}{8}$

$h(2) = \underbrace{h(4)}_{=1}\cdot\frac{5}{7} + \underbrace{h(6)}_{0}\cdot\frac{2}{7} = \frac{5}{7}$

$h(3) = 0$ (A has dropped out)

Hence: $h(0) = \frac{5}{8}\cdot\frac{15}{31} + \frac{5}{7}\cdot\frac{10}{31} + 0 = \underline{0.533}$

Back to theory:

$$h(i) = \sum u(i,j)\,h(j) \qquad i \in C \qquad u(i,j) = \frac{q(i,j)}{\lambda_i}$$

$h(i) = 1$ for $i \in A$

$h(i) = 0$ for $i \in B$.

$$h(i) = \sum_{j \neq i} \frac{q(i,j)}{\lambda_i}\, h(j) \quad \Big|\ \lambda_i$$

$$\underline{Q} = \begin{bmatrix} -\lambda_1 & q_{12} & q_{13} \\ & -\lambda_2 & \\ & & \end{bmatrix}$$

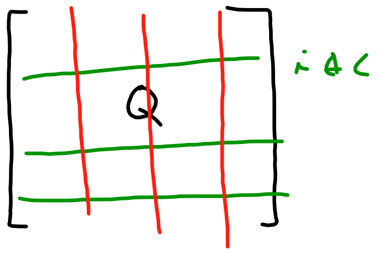
$$\lambda_i h(i) = \sum q(i,j)\, h(j)$$

$$-Q_{ii}\, h(i) = \sum_{j \neq i} Q(i,j)\, h(j)$$

$$0 = \sum_{j \in S} Q(i,j)\, h(j) \quad \text{for } i \in C$$

R is the part of Q that only consists of rows and columns that correspond to elements in C

R =



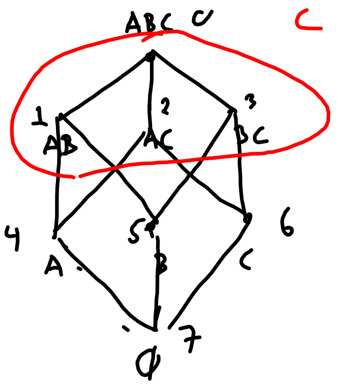
$$i \in C \qquad 0 = \sum_j Q_{ij}\, h(j) = \sum_{j \in C} R_{ij}\, h(j) + \sum_{j \in A} Q_{ij}$$

$$0 = Rh + v \qquad v = \begin{pmatrix} \sum_{j \in A} q_{1j} \\ \vdots \end{pmatrix}$$

$$(-R)h = v \quad \Big|\ (-R)^{-1}$$

$$\boxed{h = (-R)^{-1} v}$$

Back to "Hangman" example



$$Q = \begin{array}{c} \begin{array}{cccccccc} & R & & & v & & & \end{array} \\ \left[\begin{array}{cccc|cccc} -\frac{31}{30} & \frac{1}{2} & \frac{1}{3} & \frac{1}{5} & 0 & 0 & 0 & 0 \\ 0 & -\frac{8}{15} & 0 & 0 & \frac{1}{3} & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -\frac{7}{10} & 0 & \frac{1}{2} & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & -\frac{5}{6} & 0 & \frac{1}{2} & \frac{1}{3} & 0 \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}\right] \end{array}$$

$$R = \begin{bmatrix} -\frac{31}{30} & \frac{1}{2} & \frac{1}{3} & \frac{1}{5} \\ 0 & -\frac{8}{15} & 0 & 0 \\ 0 & 0 & -\frac{7}{10} & 0 \\ 0 & 0 & 0 & -\frac{5}{6} \end{bmatrix}$$

$$(-R)^{-1} = \begin{bmatrix} \frac{30}{31} & \frac{225}{248} & \frac{100}{217} & \frac{36}{155} \\ 0 & \frac{15}{8} & 0 & 0 \\ 0 & 0 & \frac{10}{7} & 0 \\ 0 & 0 & 0 & \frac{6}{5} \end{bmatrix}$$

$$h = (-R)^{-1} v = \begin{bmatrix} \frac{30}{31} & \frac{225}{248} & \frac{100}{217} & \frac{36}{155} \\ 0 & \frac{15}{8} & 0 & 0 \\ 0 & 0 & \frac{10}{7} & 0 \\ 0 & 0 & 0 & \frac{6}{5} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{3} \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.533 \\ \frac{5}{8} \\ \frac{5}{7} \\ 0 \end{bmatrix} = \begin{matrix} h(0) \\ h(1) \\ h(2) \\ h(3) \end{matrix}$$