Renewal Processes

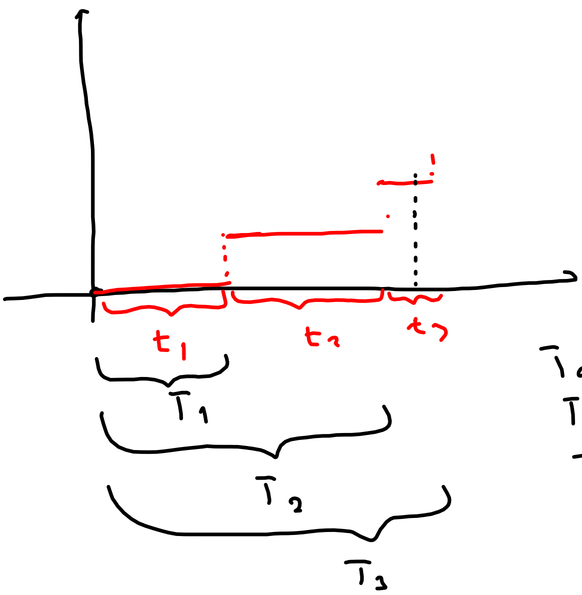
Reminder:

Law of Large numbers: Assume that $X_1, X_2, \dots$ are i.i.d random variables with finite expectation μ. If $S_n = X_1 + X_2 + \dots + X_n$, then

$$\frac{S_n}{n} = \frac{X_1 + \dots + X_n}{n} \to \mu \quad \text{with prob. } 1.$$

What is a renewal process?

Counting process



Assuming that $t_1, t_2, \dots$ are i.i.d random variables such that $P[t_i > 0] > 0$.

$T_0 = 0$
$T_1 = t_1$
$T_2 = t_1 + t_2$
$\vdots$
$T_n = t_1 + t_2 + \dots + t_n$

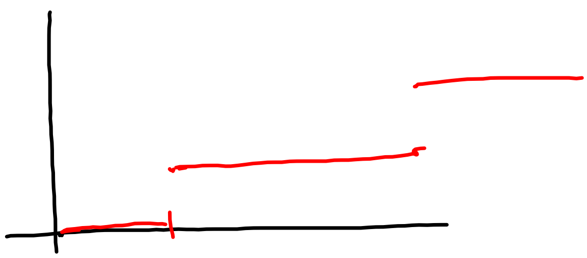
arrival times

Renewal process: $N(t) = \max\{n : T_n \le t\}$

Example: If the t_i's are λ-exponential, then N is a Poisson process with rate λ.

Example: X is a Markov chain, x be a recurrent state.

$N(t)$ is the number of returns to x before time t.



Theorem: With prob. 1,

$$\frac{N(t)}{t} \to \frac{1}{E[t_i]}$$

Proof: By the LLN

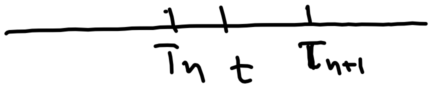
$$\frac{T_n}{n} = \frac{t_1+t_2+\cdots+t_N}{n} \to E[t_i] \quad \text{with prob 1,}$$

hence

$$\frac{n}{T_n} \to \frac{1}{E[t_i]}$$

This means that

$$\frac{N(T_n)}{T_n} \to \frac{1}{E[t_i]}$$



$N(t) = N(T_n)$

$T_n < t < T_{n+1}$

$$\frac{N(T_n)}{T_n} \geq \frac{N(t)}{t} \geq \frac{N(T_n)}{T_{n+1}}$$

$$\frac{N(T_n)}{T_n} \to \frac{1}{E[t_i]} \qquad \frac{N(t)}{t} \to \frac{1}{E[t_i]}$$

$$= \underbrace{\frac{N(T_{n+1})}{T_{n+1}}}_{\to \frac{1}{E[t_i]}} \underbrace{\left(\frac{N(T_n)}{N(T_{n+1})}\right)}_{\to 1} \to \frac{1}{E[t_i]}$$

Heuristics:

$$\frac{N(t)}{t} \to \frac{1}{E(t_i)}$$

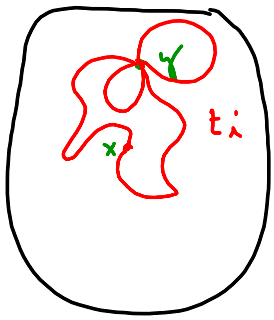
i.e. $\underline{t} \sim \underbrace{N(t)}\underbrace{E(t_i)}$

Reward function

After each period, the process gets a (stochastic) reward r_i

$(r_1, t_1), (r_2, t_2), (r_3, t_3), \dots$ are i.i.d (but r_i may depend on t_i)

Example:



r_i the number of visits to y between T_{i-1} and T_i

$$R(t) = \sum_{i=1}^{N(t)} r_i$$

Theorem: $\frac{R(t)}{t} \longrightarrow \frac{E[r_i]}{E[t_i]}$

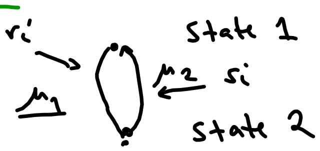
$E[r_i]$

$\uparrow$ LLN

Proof: $\frac{R(t)}{t} = \frac{1}{t}\sum_{i=1}^{N(t)} r_i = \underbrace{\frac{N(t)}{t}}_{\downarrow \ \frac{1}{E[t_i]}} \underbrace{\frac{1}{N(t)} \sum_{i=1}^{N(t)} r_i}_{} \longrightarrow \frac{1}{E[t_i]} E[r_i]$

$= \frac{E[r_i]}{E[t_i]}$

Application to two state processes:



Renewal process: $N(t)$ is the number of returns to 1 by time t.

$t_i = r_i + s_i$

Reward function: r_i

$$R(t) = \sum_{i=1}^{N(t)} r_i \quad \text{time spent in state } i \text{ up to time } t.$$

Portion of time in state 1:

$$\frac{R(t)}{t} \rightarrow \frac{E[r_i]}{E[t_i]} = \frac{\mu_1}{E[r_i] + E[s_i]} = \frac{\mu_1}{\mu_1 + \mu_2} \quad \text{with prob } 1.$$

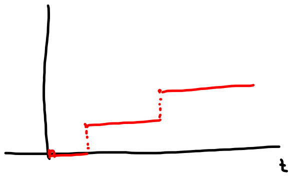
Example: Machine to produce goods.

Cost: K millions

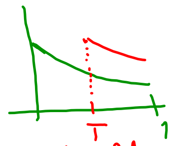
Breakdown: Uniformly distributed on $[0,1]$ years

Productivity: $e^{-\lambda t}$ million/year.

$N(t)$ = the number of replacements up to time t (renewal process)



Original model

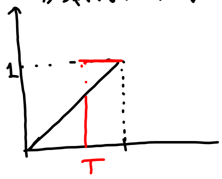


Modified model

what is the best T?

Reward process $\underbrace{-K}_{\text{cost}} + \underbrace{\int_0^{t_1} e^{-\lambda t}\,dt}_{\text{income}}$.

Distribution of the t_i's



$$E[t_i] = \int_0^T t\,dt + T\,\underbrace{P[t_i > T]}_{1-T}$$

$$= \frac{1}{2}T^2 + T(1-T) = \frac{1}{2}T^2 + T - T^2$$

$$= T - \frac{1}{2}T^2$$

We want to maximize the long term profit.

$$\frac{R(t)}{t} \longrightarrow \frac{E[r_i]}{E[t_i]} = \frac{E[r_i]}{T - \frac{1}{2}T^2}$$

What is $E[r_i]$? If a machine breaks down at time t, it has produced a profit of:

$$\int_0^t e^{-\lambda s}\,ds = \frac{1}{\lambda}\left[1 - e^{-\lambda t}\right]$$

$$\downarrow$$

$$\frac{e^{-\lambda s}}{-\lambda}$$

income from machines that die a natural death.

Hence

$$E[r_i] = -K + \underbrace{\int_0^T \frac{1}{\lambda}\left[1-e^{-\lambda t}\right]dt}_{\text{income from machines that die a natural death}} + \frac{1}{\lambda}\left[1-e^{-\lambda T}\right]\overbrace{P[t_i > T]}^{1-T}$$

$$= -K + \frac{T}{\lambda} - \frac{1}{\lambda^2}\left[1-e^{-\lambda T}\right] + \frac{1}{\lambda}\left(1-e^{-\lambda T}\right)(1-T)$$

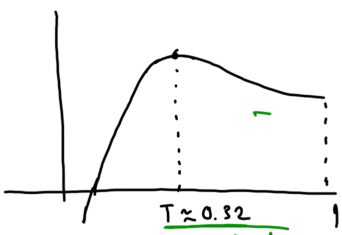
$$= -K + \frac{T}{\lambda} + \frac{1}{\lambda}\left(1-e^{-\lambda T}\right)\left(1 - T - \frac{1}{\lambda}\right)$$

Want to maximize:

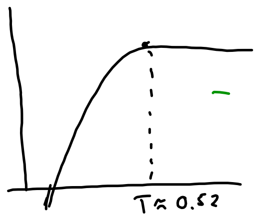
$$f(T) = \frac{E[r_i]}{E[t_i]} = \frac{-K + \frac{T}{\lambda} + \frac{1}{\lambda}\left(1-e^{-\lambda T}\right)\left(1-T-\frac{1}{\lambda}\right)}{T - \frac{1}{2}T^2} \quad \text{maximize}$$

Pictures: $K = 0.05$ (50.000 dollars) $K = 0.1$

$\lambda = 1.5$ $\lambda = 1.5$

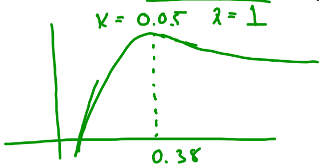


$T \approx 0.32$



$T \approx 0.52$

$K = 0.05$ $\lambda = 1$



0.38

Applications to queueing theory

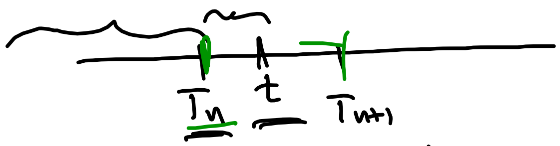
One server queue

server uses a stochastic time with mean $\frac{1}{\mu}$ ← "rate"
to serve each customer

new customers arrive with a mean of $\frac{1}{\lambda}$.

Theorem: Assume that $\lambda < \mu$ (server faster than queue). Then asymptotically the server will be busy $\frac{\lambda}{\mu}$ of the time. Hence no matter how many people are waiting initially, the queue will eventually be empty.

Sketch of the proof: How much of the time has the server been busy up to time t? T_n is the arrival of the n-th new customer



clear the initial line; clear the final line

$$\frac{\text{Time server been busy up to time } t}{t} = \frac{Z_0 + \Delta_1 + \Delta_2 + \dots + \Delta_n - Z_n}{T_n + (t - T_n)}$$

$$= \frac{\underbrace{\frac{Z_0}{n}}_{\to 0} + \underbrace{\frac{\Delta_1+\Delta_2+\dots+\Delta_n}{n}}_{\to \frac{1}{\mu}} - \underbrace{\frac{Z_n}{n}}_{\to 0}}{\underbrace{\frac{T_n}{n}}_{\to \frac{1}{\lambda}} + \underbrace{\frac{(t-T_n)}{n}}_{\to 0}} \longrightarrow \frac{\left(0 + \frac{1}{\mu} + 0\right)}{\left(\frac{1}{\lambda} + 0\right)} = \frac{\frac{1}{\mu}}{\frac{1}{\lambda}} = \frac{\lambda}{\mu}$$

$T_n = t_1 + t_2 + \dots + t_n$