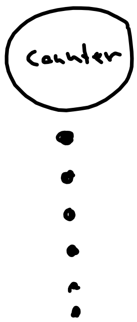
# Queueing Theory

**Last time:**




serving times $s_i$ are i.i.d with mean $\frac{1}{\mu}$

interarriving times $t_i$ are i.i.d with mean $\frac{1}{\lambda}$ ← assume $t_i$ exponential with rate $\lambda$

Poisson process.

**Theorem**: If $\lambda < \mu$, then the server will be busy $\frac{\lambda}{\mu}$ of the time.

Let $\xi_i$ be the number of new customers that join the queue while customer $i$ 's being served.

$$a_n = P[\xi_i = n] = \int_0^\infty e^{-\lambda s} \frac{(s\lambda)^n}{n!} \, dG(s)$$

where $G$ is the distribution function of $s_i$.

Hence

$$E[\xi_i] = \sum_{n=0}^\infty n P[\xi_i = n] = \sum_{n=1}^\infty n \int_0^\infty e^{-\lambda s} \frac{(\lambda s)^n}{n!} \, dG(s)$$

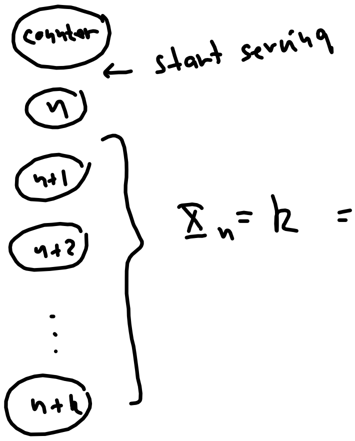
$$= \int_0^\infty e^{-\lambda s} \sum_{n=1}^\infty \frac{n}{n!} (\lambda s)^n \, dG(s)$$

$$= \int_0^\infty \lambda s \cancel{e^{-\lambda s}} \underbrace{\sum_{n=1}^\infty \frac{(\lambda s)^{n-1}}{(n-1)!}}_{\cancel{e^{\lambda s}}} \, dG(s) = \lambda \int_0^\infty s \, dG(s)$$

$$= \lambda E[s_i] = \frac{\lambda}{\mu}$$

## Embedded Markov Chain

$X_n$ - Markov chain




$X_n = k$ = number of customers behind $n$ when she is starting to be served.

### Transition probabilities:

$i > 0$ : $p_{i,i-1} = a_0$ , $p_{i,i} = a_1$ , $p_{i,i+1} = a_2$ , ....

$i = 0$ : $p_{0,0} = a_0 + a_1$ , $p_{0,1} = \underline{a_2}$ , $p_{0,2} = \underline{a_3}$ , ..

Transition matrix

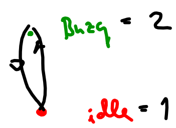
$$P = \begin{pmatrix} a_0 + a_1 , & a_2 , & a_3 , & \cdots \\ a_0 & a_1 & a_2 & a_3 \; .. \\ 0 & a_0 & a_1 & a_2 \\ 0 & 0 & a_0 & a_1 \; .. \\ & \vdots & & \end{pmatrix}$$

**Theorem:** (i) If $\lambda < \mu$, then the Markov chain is positive recurrent

(ii) If $\lambda = \mu$ , — " —— is null recurrent.

(iii) $\lambda > \mu$ , " — " — is transient.

Assume $\lambda < \mu$:


$$\frac{R(t)}{t} \longrightarrow \frac{\mu_1}{\mu_1 + \mu_2}$$

$\mu_1 \to \frac{1}{\lambda}$, $\quad \mu_1 = \frac{1}{\lambda}$, $\quad \mu_2 = E[B]$, $\quad \frac{R(t)}{t} \to 1 - \frac{\lambda}{\mu}$

$$1 - \frac{\lambda}{\mu} = \frac{\frac{1}{\lambda}}{\frac{1}{\lambda} + E[B]}$$

$$\frac{1}{\lambda} + E[B] = \frac{\frac{1}{\lambda}}{1 - \frac{\lambda}{\mu}} \Rightarrow$$

$$E[B] = \frac{\frac{1}{\lambda}}{1 - \frac{\lambda}{\mu}} - \frac{\frac{1}{\lambda}\left(\cancel{1} - \frac{\lambda}{\mu}\right)}{1 - \frac{\lambda}{\mu}} = \frac{\frac{1}{\cancel{\lambda}} \cdot \frac{\cancel{\lambda}}{\mu}}{1 - \frac{\lambda}{\mu}} = \frac{\frac{1}{\mu}}{1 - \frac{\lambda}{\mu}} = \frac{1}{\mu - \lambda}$$

The expected length of a
- idle period $\frac{1}{\lambda}$
- busy —"— $\frac{1}{\mu - \lambda}$.

**Pollaczek-Khintchine formula** The long time average waiting time is

$$W_Q = \frac{\frac{\lambda}{2} E[s_i^2]}{1 - \lambda E[s_i]}$$

**Example**: $s_i$ has mean 5 ($\mu = \frac{1}{5}$) and standard deviation 7
$t_i$ is exponential with rate $\frac{1}{6} = \lambda$

(i) Average idle period: $\frac{1}{\lambda} = 6$

Average busy period: $\frac{1}{\mu - \lambda} = \frac{1}{\frac{1}{5} - \frac{1}{6}} = \frac{1}{\frac{6}{30} - \frac{5}{30}} = \frac{1}{\frac{1}{30}} = 30$

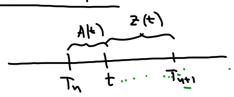
(ii) Average waiting time:

$$W_Q = \frac{\frac{\lambda}{2} E[s_i^2]}{1 - \lambda E[s_i]} = \frac{\frac{1}{2}\left(Var(s_i) + E(s_i)^2\right)}{1 - \frac{1}{6} \cdot 5}$$

$$= \frac{\frac{1}{12} \cdot 12 \left(7^2 + 5^2\right)}{\frac{1}{6} \cdot 12} = \frac{\overbrace{49 + 25}^{74}}{2} = 37$$

## Age and residual life

**Common situation**




### Discrete case

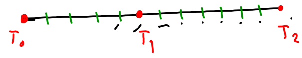
Jump only at integers; i.e. $t_i$ take values in $\{0,1,2,\dots\}$

$f_k = P[t_i = k]$

**Define**:

$A_n = \{n - T_k$ : where $T_k$ is the last jump at $n$ or before $n\}$

$Z_n = \{T_k - n$ : —— " —— first jump at or after $n\}$




A: 0 1 2 3 4 0 1 2 3 4 5 6 0 . .

Z: 0 4 3 2 1 0 6 5 4 3 2 1 0 .

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} P[A_n = i] = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} P[Z_n = i]$$

$Z$ is a Markov chain with trans. prob.

$i \neq 0$, $p_{i,i-1} = 1$

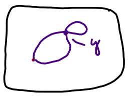
$i = 0$: $p_{0,k} = f_{k+1}$,


**State space**: If $f_k > 0$ for infinitely many $k$, $S = \{0,1,\dots\}$ irreducible.

If $K$ is the largest value such that $f_K > 0$, then $S = \underbrace{\{0,1,\dots,K-1\}}_{\text{closed and irreducible}}$.

**Stationary measure**: Cycle trick:

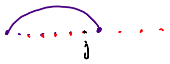
$$\mu_x(y) = E_x\Big[\sum_{n\geq 1}^{T_x - 1} \mathbb{1}_{\{y\}}(X_n)\Big] = \text{the expected number of visits to } y \text{ on the first cycle from } x \text{ to } x.$$




In this case $x = 0$:

$\mu_0(j) = P[t_i > j]$

Total mass




$$\sum \mu_0(j) = \sum_{j=0}^{\infty} P[t_i > j] = E[t_i]$$

If $E[t_i] < \infty$, we can define a stationary measure by

$$\pi(j) = \frac{\mu_0(j)}{E[t_i]} = \frac{P[t_i > j]}{E[t_i]}$$

**Ergodic theorem**: If recurrent and a s.d. exists, then

$$\frac{1}{N}\sum_{n=1}^{N} f(X_n) \longrightarrow \sum_{i\in S} f(i)\,\pi_i$$

Apply E.T. to $f = \mathbb{1}_j$ $\left( f(i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}\right.$.

Hence

$$\frac{1}{N}\sum_{n=1}^{N} \mathbb{1}_j(Z_n) \to \sum_{i\in S} f(i)\,\pi(i) = \pi(j)$$

**Convergence theorem**: If recurrent, has a std dist, and is aperiodic, then $P_0[Z_n = i] \to \pi(i)$

**When is $Z$ aperiodic?** Let $I = \{k : f_k > 0\}$. If the largest common factor of all elements in $I$ is 1, then $Z$ is aperiodic

**Theorem**: Assume that the largest common factor of element in $I$ is 1, then

$$P[Z_n = j] \longrightarrow \pi(j) = \frac{P[t_i > j]}{E[t_i]}$$

### Continuous case

**Theorem**: $$\lim \frac{1}{t}\int_0^t \mathbb{1}_{[Z_s > x]}\, ds = \frac{1}{E[t_i]}\int_x^{\infty} P[t > z]\, dz$$