Poisson processes

Def: N taking values in $\{0,1,2,\dots\}$ is a Poisson with rate λ if

$$P[N=n] = e^{-\lambda t}\frac{\lambda^n}{n!}$$

Mean: $E[N=\lambda]$

Variance: $\mathrm{Var}(N) = \lambda$.

Thm: If $N_1, N_2, \dots, N_n$ are Poisson r.v. with rates $\lambda_1, \lambda_2, \dots, \lambda_n$, then

$$N = N_1 + N_2 + \dots + N_n$$

is Poisson with rate $\lambda = \lambda_1 + \lambda_2 + \dots + \lambda_n$

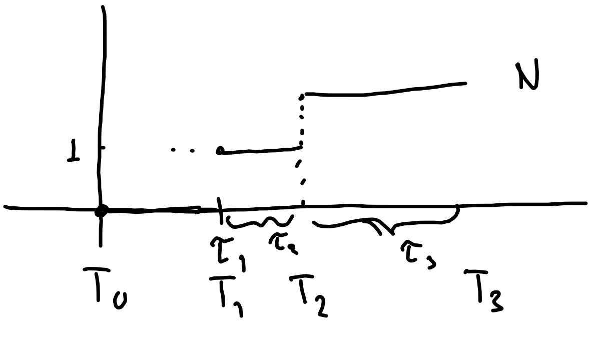
Def: A stochastic processes $\{N_t\}_{t\in[0,\infty)}$ is a Poisson process with rate λ if

(i) $N(0) = 0$

(ii) N has independent increments

(iii) $N(t) - N(s)$ is Poisson distr. with rate $\lambda(t-s)$.

Dynamical picture: τ_1, τ_2, τ_3 are independent, exponential r.v. with rate λ:



$$T_n = \tau_1 + \tau_2 + \dots + \tau_n \qquad \text{(n-th jump)}$$

$$N(t,\omega) = \max\{n : T_n \le t\}$$

Mean: $E[N(t) - N(s)] = \lambda(t-s)$

$$\mathrm{Var}[N(t) - N(s)] = \lambda(t-s)$$

Nonhomogeneous Poisson processes

Rate function: $\lambda(t)$

Mean value function: $m(t) = \int_0^t \lambda(s)\,ds$

Def: N is a nonhom. Poisson process with rate function λ if

(i) $N(0) = 0$

(ii) N has independent increments

(iii) $N(t) - N(s)$ is Poisson distributed with rate $m(t) - m(s) = \int_s^t \lambda(r)\,dr$

Infinitesimal description:

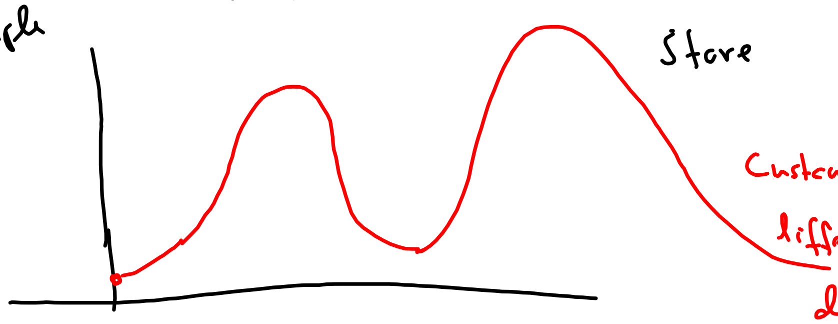
(i) $N(0) = 0$

(ii) N has ind. increments

(iii) $P[N(t+h) - N(t) = 1] = \lambda(t)h + o(h)$

(iv) $P[N(t+h) - N(t) > 1] = o(h)$

Example



Customers arriving at different rates at different times

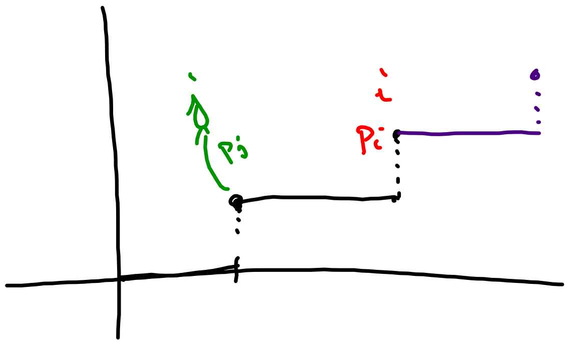
Operations on Poisson process

Adding: $N_1, N_2, \dots, N_n$ independent Poisson process with rates $\lambda_1, \lambda_2, \dots, \lambda_n$, then

$$N(t) = N_1(t) + N_2(t) + \dots + N_n(t)$$

is a Poisson proc with rate $\lambda_1 + \lambda_2 + \dots + \lambda_n$.

Thinning: Categories: $1, 2, \dots, k$. With probability p_k the jump belongs to category k.



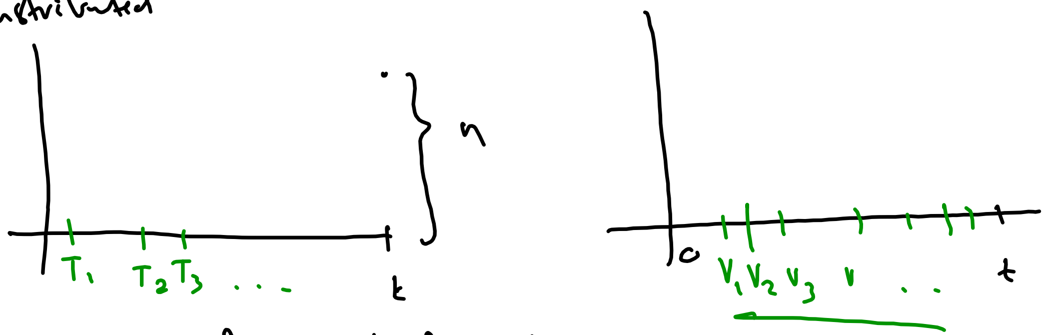
$\underline{N_j(t)}$ = number of jumps before of type j

Poisson process with rate $p_j\lambda$.

Example: Cinema
- children $\to$ 0.4
- adults $\to$ 0.6

$N(t)$ with rate λ
- $\nearrow$ N_c poisson with rate 0.4λ
- $\searrow$ N_a — " — 0.6λ.

Conditioning: If we know that $N(t) = n$, how are $T_1, T_2, \dots, T_n$ distributed



Choose n random points from the interval and order them $V_1 < V_2 < \dots < V_n$. Then $(V_1, V_2, \dots, V_n)$ has the same distribution as $(T_1, T_2, \dots, T_n)$.

Compound Poisson Processes

Poisson process N

$Y_1, Y_2, Y_3, \ldots$ are iid and independent of N.

$$S(t) = Y_1(t) + Y_2(t) + \cdots + Y_{N(t)}$$

Example: Car insurance company:

Claims come as Poisson process N.

Y_i is the random claim for accident no i.

$$S(t) = Y_1(t) + \cdots + Y_{N(t)}(t) \leftarrow \text{total claims by time } t.$$

Theorem: If N is a Poisson random variable with rate λ and $Y_1, Y_2, \ldots$ are iid and independent of N. If

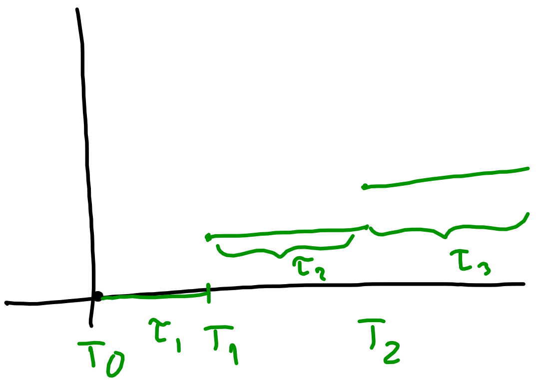
$$S = Y_1 + Y_2 + \cdots + Y_N,$$

then

$$E[S] = E[N]\,E[Y_i]$$

$$\operatorname{var}(S) = \lambda\, E[Y_i^2]$$

Renewal process



Fix i.i.d random variable $E[\tau_i] < \infty$, and make the process jump as pictured below.

$N(t) = \max\{n : T_n \le t\}$

Thm: If $P[\tau_i > 0] > 0$ and $\mu = E[\tau_i] < \infty$, then

$$\frac{N(t)}{t} \to \frac{1}{\mu}. \quad \text{with prob. } 1$$

Why $\mu N(t) \sim t$

Reward process: For each jump there is a reward r_i (i.i.d).

$$R(t) = \sum_{i=1}^{N(t)} r_i$$

$$\left.\frac{R(t)}{t} \to \frac{E[r_i]}{E[\tau_i]}\right\}$$

Example: Machine $\underbrace{\underbrace{\text{working}}_{s_i} - \underbrace{\text{repaired}}_{r_i}}_{\tau_i} - \underbrace{\text{working}_{s_{i+1}} - \text{repair}_{r_{i+1}}}_{\tau_{i+1}}$

Reward: s_i

$S(t) = \sum s_i$ = the period the machine working

$$\frac{S(t)}{t} \to \frac{E[s_i]}{E[\tau_i]} = \frac{E[s_i]}{E[s_i] + E[r_i]}$$

Problem 3, 2017

a) People coming in Poisson 4 per minute ← count time minutes.

Arriving exponentially with rate 4: Density $4e^{-4t}$

b) Time between lifts 5 seconds $= \frac{1}{12}$ minute.

Arrivals are Poisson with rate $\lambda \cdot t = 4 \cdot \frac{1}{12} = \frac{1}{3}$

$$P[\text{customers} = n] = e^{-\frac{1}{3}} \frac{\left(\frac{1}{3}\right)^n}{n!} = e^{-\frac{1}{3}} \frac{1}{3^n n!}.$$

c) Prob one person leaving + one left waiting

$$= P[\text{customers} = 2] = e^{-\frac{1}{3}} \frac{1}{3^2 \cdot 2} \approx$$

d) Prob of second lift leaving empty?

two ways

(1) no customers have arrived before second lift leaving

(2) one customer during first period, none during second.

$$P[\text{customer} = 0]^2 + P[\text{customer} = 1]\,P[\text{customer} = 0]$$

$$= \left(e^{-\frac{1}{3}}\right)^2 + \frac{e^{-1/3}}{3} \cdot e^{-1/3} = e^{-\frac{2}{3}}\left(1 + \frac{1}{3}\right) = \frac{4}{3}e^{-\frac{2}{3}}$$

Probl: 2009

c)

$$P[X_1 > t] = \sum_{n=0}^{\infty} P[X_1 > t \mid N(t)=n]\, \underline{P[N(t)=n]} \qquad \underbrace{\sum e^{-\lambda} \frac{\lambda^n}{n!}}_{e^{-\lambda}\sum\frac{\lambda^n}{n!} \atop e^{\lambda}} = 1$$

$$= \sum_{n=0}^{\infty} \left(\frac{t - D(t)}{\cancel{t}}\right)^n \underline{e^{-\lambda t}} \frac{(\lambda \cancel{t})^n}{n!} \qquad \underline{\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x}$$

$$= e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda (t - D(t)))^n}{n!} = e^{-\lambda t} e^{\lambda (t - D(t))}$$

$$= \underline{\underline{e^{-\lambda D(t)}}}$$

$$P[X_1 \le t] = 1 - e^{-\lambda D(t)}$$

d) Explain that

$$P[X_1 > t \mid \underbrace{N(t) = n}_{\text{has made } n \text{ jumps – uniformly distributed.}}] = \underline{\underline{\left(\frac{t - D(t)}{t}\right)^n}}$$

Look at the n=1 case:

$$P[\underline{X_1 > t} \mid \underline{N(t) = 1}] = \frac{1}{t} \int_0^t (1 - d(s))\, ds = \frac{1}{t}[t - D(t)]$$

$$= \underline{\underline{\frac{t - D(t)}{t}}}$$

What about n:

$$P[X_1 > t : N(t) = n] = \left(\frac{t - D(t)}{t}\right)^n \quad \text{by independence.}$$