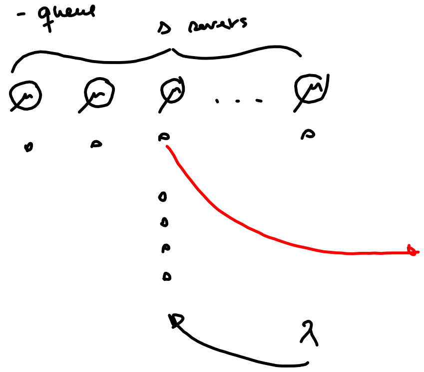
Departing customers

M/M/s - queue



$\lambda < s\mu$

π stationary distribution

$N(t)$ = the number of customers who have left before time t.

Theorem: Let X be a cont. time Markov chain started with the stationary distributed π. Fix $T>0$ (large) and let

$$Y_t = X_{T-t}.$$

Then Y is cont time Markov chain with transition prob.

$$\hat{P}_t(i,j) = \frac{\pi_j P_t(j,i)}{\pi_i}$$

Proof: Need to prove that

$$\frac{\pi_j P_t(j,i)}{\pi_i} = P[Y_{s+t}=j \mid Y_s=i \wedge Y_{s_1}=i_1 \wedge Y_{s_2}=i_2 = \dots = Y_{s_n}=i_n]$$

where $s_1, s_2, \dots, s_n < s$.

$P[B|C] = \frac{P[B\cap C]}{P[C]}$

$P(B\cap C) = P[B|C]P(C)$

We have

$$P[Y_{s+t}=j \mid Y_s=i \wedge Y_{s_1}=i_1 \wedge Y_{s_2}=i_2 = \dots = Y_{s_n}=i_n]$$

$$= \frac{P[Y_{s+t}=j \wedge Y_s=i \wedge Y_{s_1}=i_1 \wedge \dots \wedge Y_{s_n}=i_n]}{P[Y_s=i \wedge Y_{s_1}=i_1 \wedge \dots \wedge Y_{s_n}=i_n]}$$

future event A

$$= \frac{P[X_{T-(s+t)}=j \wedge X_{T-s}=i \wedge X_{T-s_1}=i_1 \wedge \dots \wedge X_{T-s_n}=i_n]}{P[X_{T-s}=i \wedge X_{T-s_1}=i_1 \wedge \dots \wedge X_{T-s_n}=i_n]}$$

future

$$= \frac{P[A \mid \cancel{X_{T-(s+t)}=j} \wedge X_{T-s}=i]\,P[X_{T-(s+t)}=j \wedge X_{T-s}=i]}{P[A \mid X_{T-s}=i]\,P[X_{T-s}=i]}$$

$$= \frac{P[X_{T-(s+t)}=j \wedge X_{T-s}=i]}{\underbrace{P[X_{T-s}=i]}_{\pi(i)}} = \frac{\pi(j)P_t(j,i)}{\pi(i)}$$

Question: I discrete time the reverse process is the same as the original process when the detailed balance equation is satisfied? Does this still hold?

Prop: Assume that π satisfies the detailed balance equation.
Then $\pi_i p_t(i,j) = \pi_j p_t(j,i)$ for all i,j and t.

Proof (almost): Detailed balance equation

$$\pi_i q_{ij} = \pi_j q_{ji} \quad \text{for all } i,j.$$

By Kolmogorov's BE:

$$p_t'(i,j) = \sum_{k\neq i} q_{ik}\, p_t(k,j) - \lambda_i\, p_t(i,j) \quad \Big|\ \pi_i$$

$$\pi_i\, p_t'(i,j) = \sum_{k\neq i} \underbrace{\pi_i q_{ik}}_{\text{Detailed balance}} p_t(k,j) - \lambda_i\, \pi_i\, p_t(i,j)$$

Notation: $y_t(i,j) = \pi_i\, p_t(i,j)$

$$\pi_i\, p_t'(i,j) = \sum_{k\neq i} \pi_k q_{ki}\, p_t(k,j) - \lambda_i\, \pi_i\, p_t(i,j)$$

$(q^T)_{ik}$

$$\to\ y_t'(i,j) = \sum_{k\neq i} q_{ki}\, y_t(k,j) - \lambda_i\, y_t(i,j)$$

$$\boxed{y_t' = Q^T y_t}$$

By Kolmogorov's FE:

$$p_t'(j,i) = \sum_{k\neq i} p_t(j,k)\, q_{ki} - \lambda_i\, p_t(j,i) \quad \Big|\ \pi_j$$

$$\boxed{z_t(i,j) = \pi_j\, p_t(j,i)}$$

$$\pi_j\, p_t'(j,i) = \sum_{k\neq i} \pi_j p_t(j,k)\, q_{ki} - \lambda_i\, \pi_j\, p_t(j,i)$$

$$z_t'(i,j) = \sum_{k\neq i} z_t(k,j)\, q_{ki} - \lambda_i\, z_t(i,j)$$

$$z_t' = Q^T z_t$$

Hence y_t and z_t satisfy the D.E. with the same initial conditions

$$y_0 = \begin{pmatrix} \pi_0 & & & \\ & \pi_1 & & \\ & & \pi_2 & \\ & & & \ddots \end{pmatrix} = z_0$$

Since the equation has a unique solution, we have $y_t = z_t$ for all t.

Hence

$$\pi_i p_t(i,j) = y_t(i,j) = z_t(i,j) = \pi_j\, p_t(j,i).$$

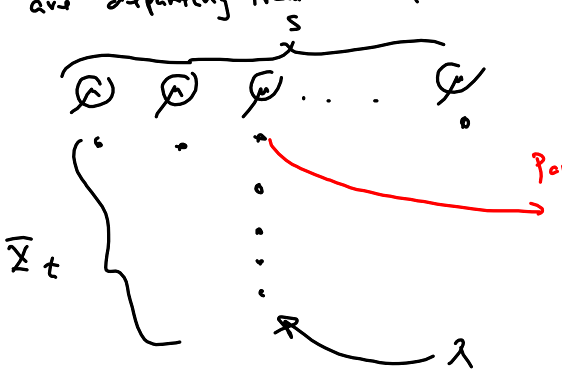
Corollary: If π satisfies the detailed balance equation, then the backward process has the same trans. prob. as the forward process; i.e.

$$\hat{p}_t(i,j) = p_t(i,j)$$

Proof:

$$\hat{p}_t(i,j) = \frac{\pi_j\, p_t(j,i)}{\pi_i} \overset{DB}{=} \frac{\cancel{\pi_i}\, p_t(i,j)}{\cancel{\pi_i}} = p_t(i,j)$$

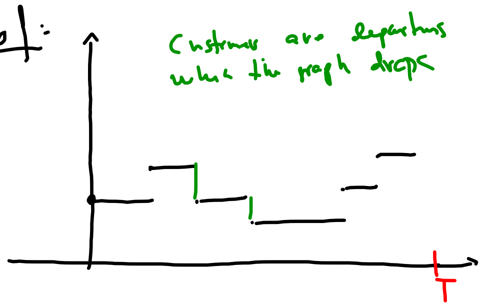
Theorem: Consider a M/M/s-queue with $\lambda < s\mu$ and assume that it is started with its stationary distribution. Then customers are departing from the system like a Poisson process with rate λ.



Poisson process rate λ. $N(t)$ = the number of customers who have departed before time t.

Proof:

Customers are departures when the graph drops



$\bar{X}(t)$

Customer leave when the backward process increases

= as often as the forward process increases

= according to Poisson rate λ.

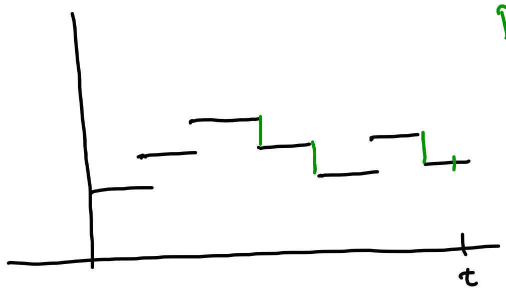
Observation: The argument still goes through even when the servers are not Markovian, provided a stationary distribution exists.

Theorem: $\bar{X}(t)$ and $N(t)$ are independent

customers currently in queue

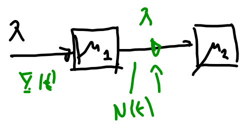
customer who has left

Proof:



Look at the backward process. Then $N(t)$ is counting how many customers that will arrive in the future, and that is independent of $\bar{X}_t$ since customer will arrive at rate λ independent of what $\bar{X}_t$ is.

Systems of queues (sect. 4.6)



equilibrium distribution

Need $\lambda < \mu_1$

First and second work independently.

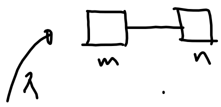
Guess for stationary distribution: Server 1: $\pi_1(m) = \frac{\mu_1 - \lambda}{\lambda}\left(\frac{\lambda}{\mu_1}\right)^m$

Server 2: $\pi_2(n) = \frac{\mu_2 - \lambda}{\lambda}\left(\frac{\lambda}{\mu_2}\right)^n$

$$\pi(m,n) = \frac{\mu_1-\lambda}{\lambda}\left(\frac{\lambda}{\mu_1}\right)^m \frac{\mu_2-\lambda}{\lambda}\left(\frac{\lambda}{\mu_2}\right)^n = c\left(\frac{\lambda}{\mu_1}\right)^m\left(\frac{\lambda}{\mu_2}\right)^n$$

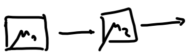
Want to check that this really is a stationary distribution:

Kind of transition:

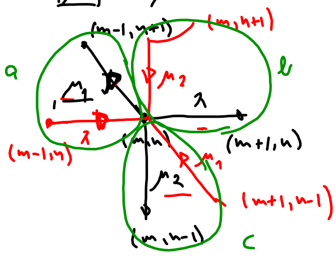


$(m,n) \to (m+1,n)$ with rate λ

$(m,n) \to (m-1, n+1)$ with rate μ_1



$(m,n) \to (m, n-1)$ with rate μ_2



Need to check that total rate out equals total rate in when $\pi(m,n) = c\left(\frac{\lambda}{\mu_1}\right)^m\left(\frac{\lambda}{\mu_2}\right)^n$

Look at a: Out $\mu_1 \pi(m,n) = \mu_1 c \left(\frac{\lambda}{\mu_1}\right)^m\left(\frac{\lambda}{\mu_2}\right)^n$

In $\lambda \pi(m-1,n) = \lambda c \left(\frac{\lambda}{\mu_1}\right)^{m-1}\left(\frac{\lambda}{\mu_2}\right)^n$

$= c\mu_1\left(\frac{\lambda}{\mu_1}\right)^m\left(\frac{\lambda}{\mu_2}\right)^n$

Look at b: Out: $\lambda c\left(\frac{\lambda}{\mu_1}\right)^m\left(\frac{\lambda}{\mu_2}\right)^n$

In: $\mu_2 c\left(\frac{\lambda}{\mu_1}\right)^m\left(\frac{\lambda}{\mu_2}\right)^{n+1} = \lambda c\left(\frac{\lambda}{\mu_1}\right)^m\left(\frac{\lambda}{\mu_2}\right)^n$

Look at c: Out: $\mu_2 c\left(\frac{\lambda}{\mu_1}\right)^m\left(\frac{\lambda}{\mu_2}\right)^n$

In: $\mu_1 c\left(\frac{\lambda}{\mu_1}\right)^{m+1}\left(\frac{\lambda}{\mu_2}\right)^{n-1} = \mu_2 c\left(\frac{\lambda}{\mu_1}\right)^m\left(\frac{\lambda}{\mu_2}\right)^n$