Cont. time Markov chains

State space: $S <$ finite / countable

Time line: $T = [0, \infty)$

Transition prob: $p_t(i,j)$

Process: $\{X_t\}_{t \in [0,\infty)}$ s.t.

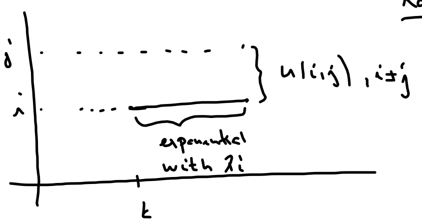
$$P[X_{s+t} = j \mid X_s = i \wedge X_{s_1} = i_1 \wedge X_{s_2} = i_2 \wedge \dots]$$

$$= p_t(i,j) \quad \text{for } s_1, s_2, \dots < s.$$

The Chapman-Kolmogorov equations

$$p_{t+s}(i,j) = \sum_{k \in S} p_s(i,k)\, p_t(k,j) \;; \text{ i.e. } \; P_{t+s} = P_s P_t$$

Dynamical description:



Rates: $q_{ij} = \lambda_i u(i,j) \quad i \neq j$

$$\sum_{j \neq i} q_{ij} = \sum_{j \neq i} \lambda_i u(i,j)$$

$$= \lambda_i \sum_{j \neq i} u(i,j) = \lambda_i$$

$$u(i,j) = \frac{q_{ij}}{\lambda_i}$$

How to translate between the two descriptions?

$$q_{ij} = \lim_{h \to 0} \frac{p_{ij}(h)}{h} = \lim_{h \to 0} \frac{p_{ij}(h) - p_{ij}(0)}{h} = p_{ij}'(0)$$

Opposite direction: know q_{ij}, how do I find $p_{ij}(t)$?

$$R = Q = \begin{bmatrix} -\lambda_1 & q_{01} & q_{02} & \\ q_{10} & -\lambda_2 & q_{12} & \\ & & \ddots & \\ & & & \ddots \end{bmatrix} \quad \text{"infinitesimal generator"}$$

Kolmogorov's backward equation:

Matrix form $\quad P_t' = QP_t$

Component form

$$p_t'(i,j) = \sum_{k \neq i} q_{ik}\, p_t(k,j) - \lambda_i p_t(i,j)$$

(set of equations with the same second component)

Kolmogorov's forward equation:

Matrix form $\quad P_t' = P_t Q$

Component form:

$$p_t'(i,j) = \sum_{k \neq j} p_t(i,k)\, q_{kj} - \lambda_j p_t(i,j)$$

(set of equations with the same first component)

Stationary distribution

Distribution: $\pi = (\pi_0, \pi_1, \pi_2, \dots)$ $\quad 0 \le \pi_i \le 1$, $\quad \sum_{i \in S} \pi_i = 1$

$\pi_t(j)$ = prob that at time t the process is in j given that it is started with distribution π.

$$\pi_t(j) = \sum_{i \in S} \pi(i) p_t(i,j) = (\pi p_t)_j$$

$$\pi_t = \pi p_t$$

We call π a stationary distribution if $\pi = \pi_t = \pi p_t$ for all t.

Theorem: π is stationary iff $\pi Q = 0$.

Proof: (i) Assume that π is stationary, i.e. $\pi p_t = \pi$ for all t.

$$0 = \frac{d}{dt}(\pi p_t) = \pi p_t' \overset{\text{Forward Kolmogorov}}{=} \underbrace{\pi p_t}_{\pi} Q = \pi Q.$$

(ii) Assume that $\pi Q = 0$. Then

$$\frac{d}{dt}(\pi p_t) = \pi p_t' \overset{\text{Backward Kolmogorov}}{=} \pi Q p_t = 0$$

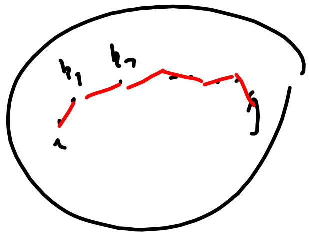
Hence πp_t is constant, i.e. $\pi p_t = \pi p_0 = \pi$.

Convergence

Def: X is irreducible if for any two states i and j, there is a chain of states such that

$$i = k_0, k_1, k_2, \dots, k_n = j$$

and $q(k_{i-1}, k_i) > 0$



Lemma: If X is irreducible, then $P_t(i,j) > 0$ for all i,j and all $t > 0$.

Theorem: If X is irreducible and has a stationary distribution π, then $P_t(i,j) \to \pi(j)$

The detailed balance equation

A distribution π satisfies the detailed balance equation if for all i,j,

$$\pi_i q_{ij} = \pi_j q_{ji}$$

(it is then automatically stationary)

Backward process

Start X with the stationary distr. and let

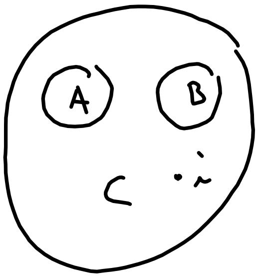
$$Y_s = X_{T-s} \quad (T \text{ fixed})$$

Then Y is a cont. time MC with stat. dist. π and

$$\hat{P}_t(i,j) = \frac{\pi_j P_t(j,i)}{\pi_i}$$

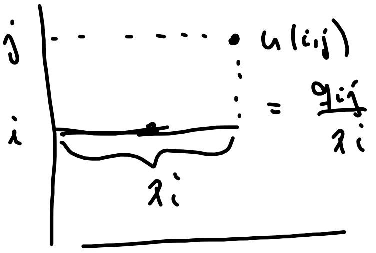
If π satisfies the detailed balance equation, $\hat{P}_t(i,j) = P_t(i,j)$)

Exit distributions



What is the prob. $h(i)$ of hitting A before B when we start in i ?

$$h(i) = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{if } i \in B \\ \sum\limits_{j \neq i} \frac{q_{ij}}{\lambda_i} h(j) & \text{if } i \in C \end{cases}$$



$$\underline{\lambda_i h(i)} = \sum_{j \neq i} q_{ij} h(j) = \sum_{\substack{j \in C \\ j \neq i}} q_{ij} \underline{h(j)} + \underbrace{\sum_{j \in A} q_{ij}}_{\underline{v_i}}$$

$$\sum_{\substack{j \in C \\ j \neq i}} q_{ij} h(j) - \lambda_i h(i) = \underline{-v_i}$$

Reduced matrix

$$R = \left(\quad Q \quad \right)$$ delete all rows and columns that do not belong to C.

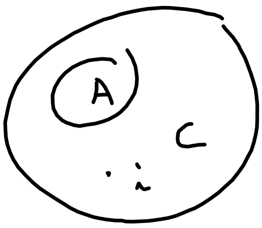
Matrix form

$$Rh = -v$$

$$(-R)h = v$$

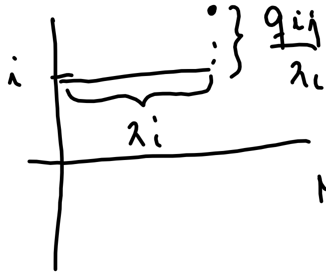
$$\underline{\underline{h = (-R)^{-1} v}}$$

Exit times



$g(i)$ = the average time to hit A when starting in i.

$$g(i) = \begin{cases} 0 & \text{if } i \in A \\ = \dfrac{1}{\lambda_i} + \displaystyle\sum_{\substack{j\in C \\ j\neq i}} g(j)\,\dfrac{q_{ij}}{\lambda_i} & \text{if } i \in C \end{cases}$$



Multiply by λ_i:

$$\lambda_i g(i) = 1 + \sum_{\substack{j\in C \\ j\neq i}} g(j)\, q_{ij}$$

i.e.

$$\underbrace{\sum_{\substack{j\in C \\ j\neq i}} q_{ij}\, g(j) - \lambda_i g(i)} = -1$$

$$Rg = -\mathbb{1} = -\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$(-Rg) = \mathbb{1}$$

$$g = (-R)^{-1}\,\mathbb{1}$$

Exam 2013, problem 2

$S = \{0,1,2\}$

Chapman-Kolmogorov

$$P_{ij}(t+s) = \sum_{k=0}^{2} p_{ik}(t)\, p_{kj}(s)$$

Proof:

condition on the situation at time t

$$P_{ij}(t+s) = P_i[X(t+s)=j] =$$

$$= \sum_{k=0}^{2} \underbrace{P_i[X(t+s)=j \mid X(t)=k]}\, P_i[X(t)=k]$$

$$= \sum_{k=0}^{2} P_{(k,j)}(s)\, P_{ik}(t)$$

$$P(t) = \begin{bmatrix} p_{00}(t) & p_{01}(t) & \cdots \\ p_{10}(t) & p_{11}(t) & \cdots \\ \vdots & \vdots & \end{bmatrix}$$ Sums of rows are all 1

$$P'(t) = \begin{bmatrix} p'_{00}(t) & p'_{01}(t) & \cdots \\ p'_{10}(t) & p'_{11}(t) & \\ \vdots & \vdots & \end{bmatrix}$$ Sums of al rows are 0.

Infinitesimal generator $R = P'(0)$.

Want to prove that $R = P'(0)$ is Q.

Recall that $q_{ij} = p'_{ij}(0) \quad i \neq j$

thus

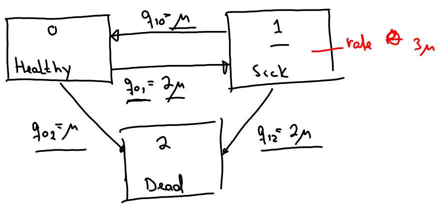
$$R = P'(0) = \begin{bmatrix} -\lambda_1 & q_{01} & q_{02} & \cdot \\ q_{10} & -\lambda_2 & q_{12} & \cdots \\ \vdots & & & \end{bmatrix} = Q$$

b) Derive the forward Kolmogorov equation: $P'_t = P_t Q$

$$P'_t = \lim_{h\to 0} \frac{P(t+h)-P(t)}{h} = \lim_{h\to 0} \frac{P(t)P(h)-P(t)\cdot I}{h}$$

$$= \lim_{h\to 0} P(t)\,\frac{P(h)-P(0)}{h} = P(t)P'(0) = P(t)Q.$$

c)



$$Q = \begin{bmatrix} -3\mu & 2\mu & \mu \\ \mu & -3\mu & 2\mu \\ 0 & 0 & 0 \end{bmatrix}$$

Give you expressions: $P_{00}(t), P_{01}(t), P_{10}(t), P_{11}(t)$

What are the other ones

$$P_{00}(t) + P_{01}(t) + P_{02}(t) = 1 \Rightarrow P_{02}(t) = 1 - \overbrace{P_{00}(t) - P_{01}(t)}^{\text{known}}$$

Similarly:

$$P_{12}(t) = 1 - P_{10}(t) - P_{11}(t)$$

$$P_{20}(t) = 0 = P_{21}(t) = 0, \quad P_{22}(t) = 1$$

e) What is the average time an individual is sick given that she is born healthy? x

is born sick? y

$$\left.\begin{aligned} x &= \frac{2}{3}y + \frac{1}{3}\cdot 0 \\ y &= \frac{1}{3\mu} + \frac{1}{3}x + \frac{2}{3}\cdot 0 \end{aligned}\right\} \quad x = \frac{2}{3}y$$

$$y = \frac{1}{3\mu} + \frac{1}{3}x = \frac{1}{3\mu} + \frac{2}{9}y$$

$$\frac{7}{9}y = \frac{1}{3\mu} \Rightarrow y = \frac{9}{7}\cdot\frac{1}{3\mu} = \frac{3}{7\mu}$$

$$x = \frac{2}{3}\cdot y = \frac{2}{3}\cdot\frac{3}{7\mu} = \frac{2}{7\mu}$$