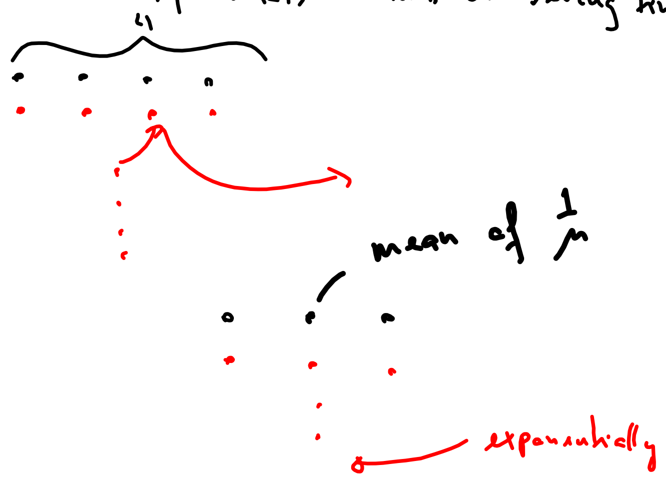
Review of queuing theory

Kendall's notation:

Nature of arrivals / Nature of the servers / number of servers

G – general arrival times/serving times given by distribution with means $\frac{1}{\lambda}$ and $\frac{1}{\mu}$

M – memoryless (exponential) arrivals or serving times.



G/G/2

M/G/3

↑

G/G/1 If $\lambda < \mu$, the queue will get empty and will be busy $\frac{\lambda}{\mu}$ of the time.

M/G/1 : X_n

$\bullet \leftarrow$ customer number n

$$\left.\begin{array}{ll} n+1 & \bullet \\ n+2 & \bullet \\ & \vdots \\ n+k & \bullet \end{array}\right\} X_n = k$$

If $\lambda > \mu$, X_n is transient.

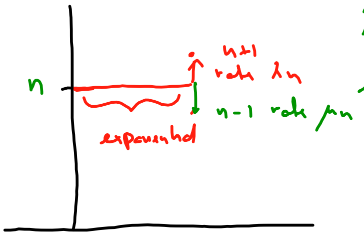
If $\lambda = \mu$, X_n is null recurrent.

If $\lambda < \mu$, X is positive recurrent

M/M/s - queues.

Birth and death processes.

Cont time Markov chain X_t with state space $S=\{0,1,2,\dots\}$



λ_n is rate of births (or arrivals)

μ_n is rate of deaths (or finished serving)

Stationary distribution: Detailed balance distribution π

$$\pi(n)\mu_n = \pi(n-1)\lambda_{n-1}$$

$$\pi(n) = \frac{\lambda_{n-1}}{\mu_n}\pi(n-1)$$

Inductively: $\pi(n) = \dfrac{\lambda_0\lambda_1\dots\lambda_{n-1}}{\mu_1\mu_2\dots\mu_n}\pi(0)$

Need: $\sum_{n=0}^{\infty}\pi(n) = 1$, i.e.

$$1 = \pi(0) + \sum \frac{\lambda_0\lambda_1\dots\lambda_{n-1}}{\mu_1\mu_2\dots\mu_n}\pi_0 = \underline{\left(1+\sum_{n=1}^{\infty}\frac{\lambda_0\lambda_1\dots\lambda_{n-1}}{\mu_1\mu_2\dots\mu_n}\right)\pi_0}$$

Assuming that $\sum_{n=1}^{\infty}\dfrac{\lambda_0\dots\lambda_{n-1}}{\mu_1\dots\mu_n} < \infty$, we get

$$\pi_0 = \frac{1}{\left(1+\sum_{n=1}^{\infty}\frac{\lambda_0\lambda_1\dots\lambda_{n-1}}{\mu_1\mu_2\dots\mu_n}\right)}$$

$$\pi_n = \frac{\lambda_0\dots\lambda_{n-1}}{\mu_1\dots\mu_n}\cdot\frac{1}{\left(1+\sum_{n=1}^{\infty}\frac{\lambda_0\lambda_1\dots\lambda_{n-1}}{\mu_1\mu_2\dots\mu_n}\right)}$$

The constant situation: $\lambda_n = \lambda$, $\mu_n = \mu$

Then
$$1+\sum_{n=1}^{\infty}\frac{\lambda_0\dots\lambda_{n-1}}{\mu_1\dots\mu_n} = 1+\sum_{n=1}^{\infty}\frac{\lambda^n}{\mu^n} = 1+\sum_{n=1}^{\infty}\left(\frac{\lambda}{\mu}\right)^n$$

$$= \sum_{n=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^n = \frac{1}{1-\frac{\lambda}{\mu}} = \underline{\frac{\mu}{\mu-\lambda}}$$

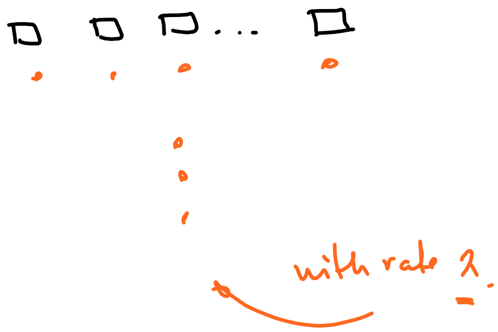
$$\sum_{n=0}^{\infty}x^n = \frac{1}{1-x}$$

Hence
$$\pi_0 = \frac{1}{\left(1+\sum_{n=1}^{\infty}\frac{\lambda_0\dots\lambda_{n-1}}{\mu_1\dots\mu_n}\right)} = \frac{1}{\frac{\mu}{\mu-\lambda}} = \frac{\mu-\lambda}{\mu} = \underline{1-\frac{\lambda}{\mu}}$$

$$\pi_n = \left(\frac{\lambda}{\mu}\right)^n\pi_0 = \left(\frac{\lambda}{\mu}\right)^n\cdot\left(1-\frac{\lambda}{\mu}\right)$$

Multiserver queues

M/M/s   s counters all with rate μ — total rate $s\mu$.



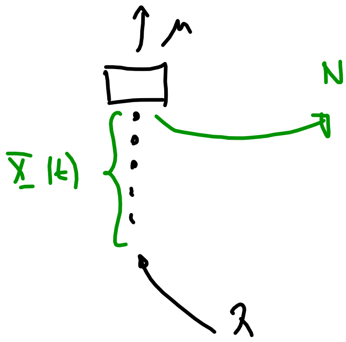
<u>Theorem</u>: If $\lambda < s\mu$, then the queues are positive recurrent

If $\lambda = s\mu$, —"— null recurrent

If $\lambda > s\mu$, the queues are transient.

Departing distributions

M/M/1 - setting

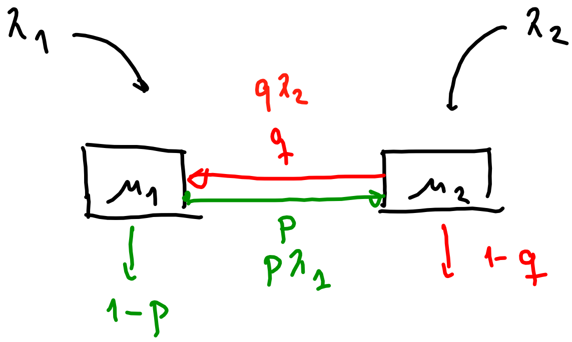


$N(t)$ - number of customers that have left up till time t

Theorem: If we start $X(t)$ with the stationary distribution, then $N(t)$ is a Poisson process with rate λ and $X(t)$ and $N(t)$ are independent.

Proof: Run the process backwards.

Systems of queues



Start both queues in stationary distribution

Total incoming process to counter 1 : $\lambda_1 + q\lambda_2$ Poisson.

———— / ———— counter 2 : $\lambda_2 + p\lambda_1$ Poisson

$$\pi(n,m) = C\left(\frac{\lambda_1 + q\lambda_2}{\mu_1}\right)^n \left(\frac{\lambda_2 + p\lambda_1}{\mu_2}\right)^m$$

customer for counter 1

customers for counter 2

Brownian motion

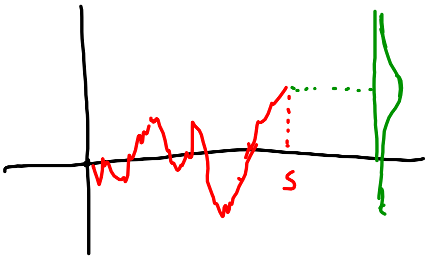
$\{B_t\}_{t\in[0,\infty)}$ $\quad S = \mathbb{R}.$

(i) $B_0 = 0$

(ii) B has independent increments

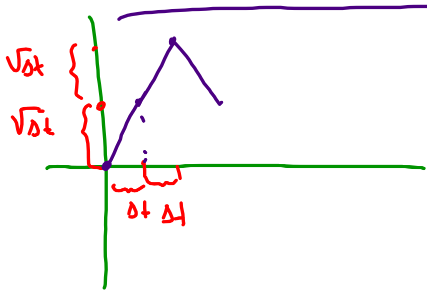
(iii) $B_t - B_s$ has a normal distribution with mean 0 and variance $t-s$.

(iv) B has continuous paths



Intuition: BM is the "limit" of random walks

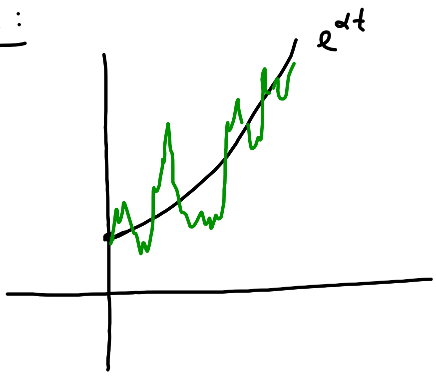
$$X(\omega,t) = \sum_{s<t} \omega(s)\sqrt{\Delta t}$$



Geometric Brownian motion:

$$X_t = e^{(\alpha - \frac{\beta^2}{2})t + \beta B_t}$$

$$E[X_t] = e^{\alpha t}$$



Standard trick: $E[B_t B_s]$ for $t > s$.

$$E[B_t B_s] = E[(B_t - B_s + B_s)B_s] = E[(B_t - B_s)(B_s - \overset{0}{\overset{\shortparallel}{B_0}})] + E[B_s^2]$$

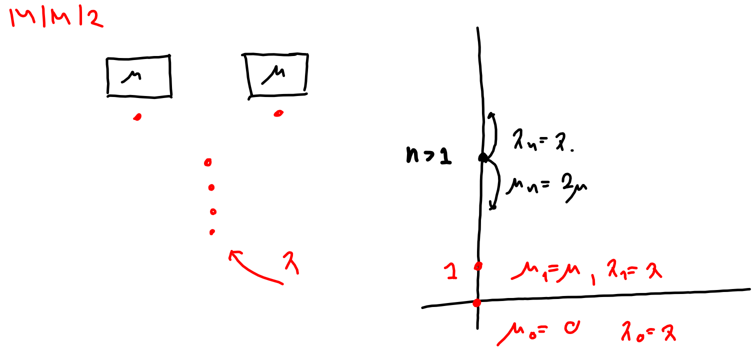
independent $\qquad$ independent

$$= \underset{0}{\underset{\shortparallel}{E[B_t - B_s]}}\,\underset{0}{\underset{\shortparallel}{E[B_s - B_0]}} + E[B_s^2] = 0 + s = s.$$

$$E[B_t B_s] = \min\{s, t\}.$$

Exam 2006, problem 2

M/M/2



c) Stationary distribution:

Detailed balance between 1 and 0.

$$\pi(1)\mu=\pi(0)\cdot\lambda \Rightarrow \underline{\pi(1)=\frac{\lambda}{\mu}\pi(0)}.$$

Detailed balance between n and $n-1$ for $n>1$

$$\pi(n)2\mu=\pi(n-1)\lambda \Rightarrow \pi(n)=\frac{\lambda}{2\mu}\pi(n-1)$$

We get

$$\pi(1)=\frac{\lambda}{\mu}\pi(0)$$

$$\pi(2)=\frac{\lambda}{2\mu}\pi(1)=\frac{\lambda}{2\mu}\cdot\frac{\lambda}{\mu}\pi(0)$$

$$\pi(3)=\frac{\lambda}{2\mu}\pi(2)=\left(\frac{\lambda}{2\mu}\right)^2\left(\frac{\lambda}{\mu}\right)\pi(0)$$

$\vdots$

$$\pi(n)=\left(\frac{\lambda}{2\mu}\right)^{n-1}\left(\frac{\lambda}{\mu}\right)\pi(0)$$

Need $1=\sum\pi(n)=\pi(0)+\frac{\lambda}{\mu}\pi(0)+\sum_{n=2}^{\infty}\left(\frac{\lambda}{2\mu}\right)^{n-1}\left(\frac{\lambda}{\mu}\right)\pi_0$

$$=\pi(0)\left(1+\frac{\lambda}{\mu}\sum_{n=1}^{\infty}\left(\frac{\lambda}{2\mu}\right)^{n-1}\right)$$

need $\frac{\lambda}{2\mu}<1$, i.e. $\lambda<2\mu$ $\qquad \sum_{n=0}^{\infty}x^n=\frac{1}{1-x}$

$$=\pi(0)\left(1+\frac{\lambda}{\mu}\frac{1\cdot 2\mu}{\left(1-\frac{\lambda}{2\mu}\right)2\mu}\right)$$

$$=\pi(0)\left(1+\frac{\lambda}{\cancel{\mu}}\frac{2\cancel{\mu}}{2\mu-\lambda}\right)=\pi(0)\left(1+\frac{2\lambda}{2\mu-\lambda}\right)$$

$$=\pi(0)\left(\frac{2\mu-\lambda}{2\mu-\lambda}+\frac{2\lambda}{2\mu-\lambda}\right)=\pi(0)\frac{2\mu+\lambda}{2\mu-\lambda}$$

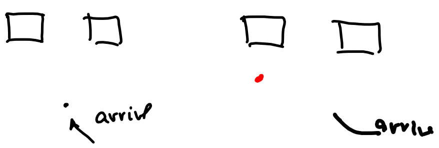
Hence

$$\pi(0)=\frac{1}{\frac{2\mu+\lambda}{2\mu-\lambda}}=\frac{2\mu-\lambda}{2\mu+\lambda}$$ provided $\lambda<2\mu$.

Consequently $\pi(n)=\left(\frac{\lambda}{2\mu}\right)^{n-1}\frac{\lambda}{\mu}\cdot\frac{2\mu-\lambda}{2\mu+\lambda}=\frac{\lambda^n}{2^{n-1}\mu^n}\frac{2\mu-\lambda}{2\mu+\lambda}.$

e) If you arrive as a new customer,

what is the prob that you will be served immediately?



$$\pi(0)+\pi(1)=\frac{2\mu-\lambda}{2\mu+\lambda}+\frac{\lambda}{\mu}\cdot\left(\frac{2\mu-\lambda}{\ldots}\right)$$

How do we compute the expectation of geometric BM?

$$E\left[e^{(\alpha-\frac{\beta^2}{2})t+\beta B_t}\right] = e^{\alpha t} \ ?$$

We have

$$E\left[e^{(\alpha-\frac{\beta^2}{2})t+\beta B_t}\right] = e^{(\alpha-\frac{\beta^2}{2})t} E\left[e^{\beta B_t}\right]$$

$$\underline{e^{(\text{---})}} \cdot e^{\beta B_t} \qquad = e^{(\alpha-\frac{\beta^2}{2})t} \cdot e^{\frac{\beta^2}{2}t} = \underline{e^{\alpha t}}$$

Look at

$$E\left[e^{\beta B_t}\right] = \int_{-\infty}^{\infty} e^{\beta x} \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}}\,dx \qquad N(0,t)$$

$$= \int_{-\infty}^{\infty} \frac{e^{-\frac{x^2-2\beta t x}{2t}}}{\sqrt{2\pi t}}\,dx = \int_{-\infty}^{\infty} \frac{e^{-\frac{\overbrace{x^2-2\beta t x+\beta^2 t^2}^{(x-\beta t)^2} - \beta^2 t^2}{2t}}}{\sqrt{2\pi t}}\,dx$$

$$= e^{-\frac{\beta^2 t}{2}} \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\beta t)^2}{2t}}}{\sqrt{2\pi t}}\,dx \qquad y = x-\beta t, \quad dy = dx$$

$$= e^{-\frac{\beta^2 t}{2}} \int_{-\infty}^{\infty} \frac{e^{-\frac{y^2}{2t}}}{\sqrt{2\pi t}}\,dy = 1 \qquad = e^{-\frac{\beta^2 t}{2}}$$