STK 2130 - 3. april 2024

(2.27) Rock concert with arrivals @ rate 50

→ 30 are female (∗)
20 are male

→ THINNING

"Main" Poisson process with rate 50 splits into two independent PP with rates 30 and 20

a) What is the probability for all 3 of the first 3 arrivals to be females?

→ recommend: turn off STK 2130 brain, it is a simple question.

From rates (∗) → 3:2 ratio of females to males.

→ chance of first guest being female $= 3/5$.

— " — second — " — $= 3/5$

EVEN GIVEN THAT 1st guest is female

(Markov property)

— " — third — " — $= 3/5$

$$\mathbb{P}(\text{first 3 guests female}) = \left(3/5\right)^3.$$

b) Same situation: arrivals to concert @ $\begin{cases} 20 \text{ males} \\ 30 \text{ females} \end{cases}$

$$\mathbb{P}\left(\text{both guests arriving in } [0,3] \;\middle|\; \text{both arrived in } [0,5]\right)$$

ONE WAY:

$$= \frac{\mathbb{P}(\{\text{both guests in } [0,3]\} \cap \{\text{both guests in } [0,5]\})}{\mathbb{P}(\text{both in } [0,5])}$$

$= \ldots$ the STK1100 way.

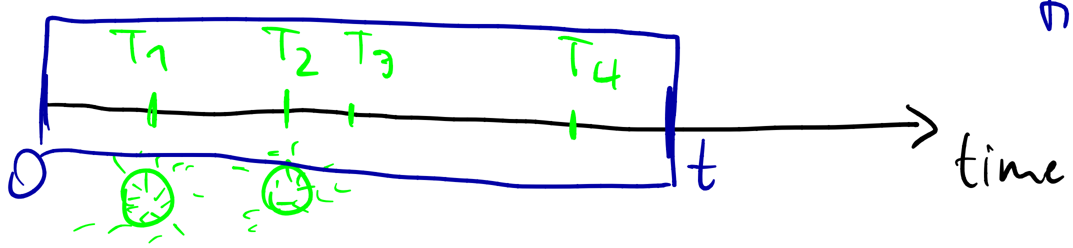
BUT:

THEOREM 2.15 (in the book)

Let $T_1, T_2, T_3, \ldots$ be arrival times of a PP and $U_1, U_2, \ldots, U_n \sim \text{Unif}[0,t]$, $\underline{U_1, U_2, \ldots U_n \text{ indep.}}$

Conditioning on $N(t) = n$, $(T_1, T_2, \ldots, T_n)$ has the same distribution as $(U_1, U_2, \ldots, U_n)$.

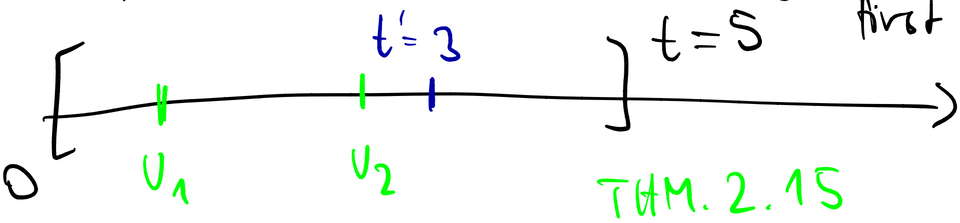
$[0,t] \to$ impose that there are 4 arrival times



b) (cont'd)

because it is given that both guest arrived in the first 5 minutes

$U_1, U_2 \sim \text{Unif}$



THM. 2.15

$$\underbrace{\mathbb{P}(U_1 \le 3,\ U_2 \le 3)}_{\substack{U_1, U_2 \text{ independent,} \\ U_1, U_2 \sim \text{Unif}([0,5])}} = \mathbb{P}(\text{both guests arriving in } [0,3] \mid \text{both arrived in } [0,5])$$

$$\overset{\text{indep.}}{=} \mathbb{P}(U_1 \le 3) \cdot \mathbb{P}(U_2 \le 3)$$

$$= \left(3/5\right) \cdot \left(3/5\right) = \underline{\underline{\left(3/5\right)^2}}.$$

c) Still in the same setting:

rock concert with PP arrival times @ rate 50.

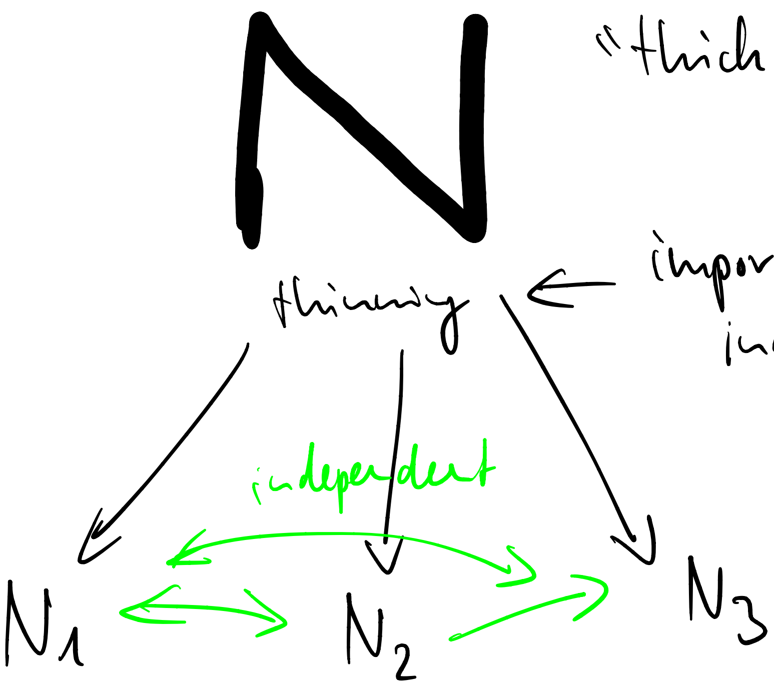
Now, instead of $50 = \begin{cases} 30 \text{ female} \\ 20 \text{ male} \end{cases}$, split as

$$50 = \begin{cases} 25 & \text{buy 1 ticket} \\ 20 & \text{buy 2 tickets} \\ 5 & \text{buy 3 tickets} \end{cases}$$

(a different way of thinning the PPP)

$N_i :=$ PPP of buying i tickets

What is distribution of (N_1, N_2, N_3)?



"thick process"

important property: independence

distribution

$\boxed{X \sim N(0,4)}$

$\mathbb{P}(X \leq 1) = \ldots$

concrete probabilities

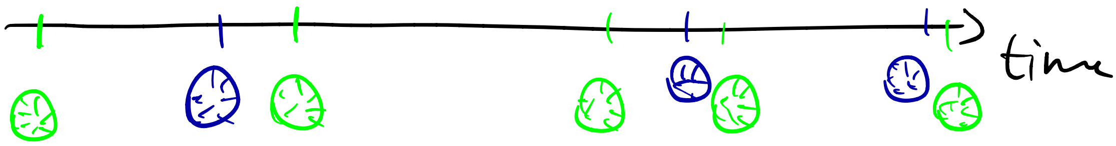
Independence of (N_1, N_2, N_3) implies that the joint distribution is simply three indep. Poisson point processes with expected values 25, 20 and 5. $(N_1, N_2, N_3) \sim (PPP_1(25), PPP_2(20), PPP_3(5))$

$\mathbb{P}(N_2(30 \text{ min}) = 4) = \ldots$

2.28

2 ways in which lightbulb can be replaced:

$$\left.\begin{array}{l} \boxed{\text{failure} \sim \exp(200)} \\ \boxed{\text{preventive replacement} \sim \exp(100)} \end{array}\right\} \text{arrival times, independently of each other}$$



$$\left.\begin{array}{l} N_{\text{failure}} \\ N_{\text{preventive replacement}} \end{array}\right\} \text{are both PPP, independent ones, with rates } 200 \text{ and } 100.$$

$$\begin{tikzcd} & N_1 \\ N_{\text{thinned}} \arrow[ur] \arrow[dr] & \\ & N_2 \end{tikzcd}$$

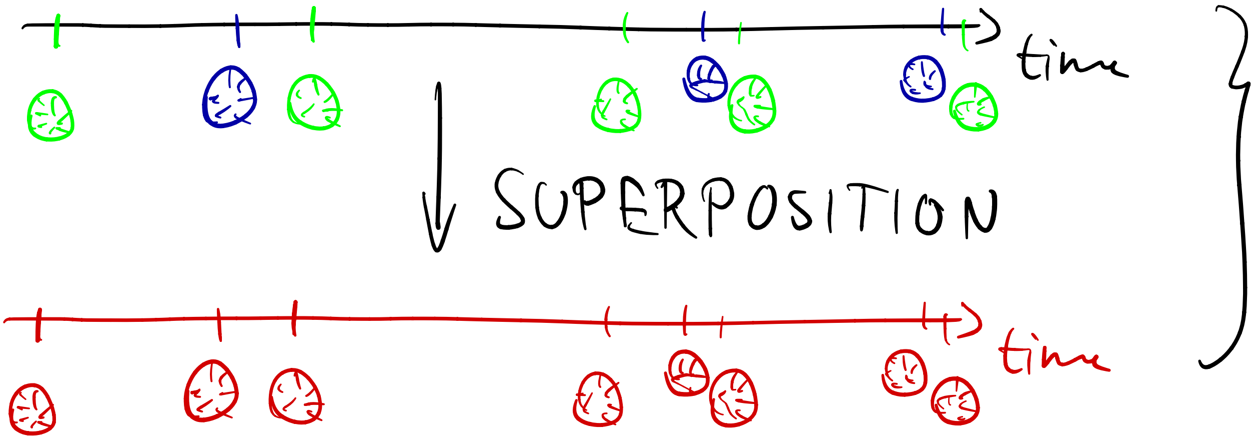
THINNING

→ prev. exercise

$$\begin{tikzcd} N_{\text{failure}} \arrow[dr] & \\ & N_{\text{total}} \\ N_{\text{preventive replacement}} \arrow[ur] & \end{tikzcd}$$

SUPERPOSITION

→ current exercise



"CLOCKS ASSUME THE SAME COLOUR"

(failure + preventive replacement) $\Longrightarrow$ replacement

<u>rates</u> are added

⚠ careful with terminology → expectations of the PPP get added

sanity check: does the expected number of events go up due to superposition (i.e. greater number of red clocks than green clocks?)

a) $N_{failure}$ has expectation $\frac{1}{200}$ (expected arrivals per day)

$N^{preventive}_{replacement}$ has expectation $\frac{1}{100}$ (expected arrivals per day)

$$\Rightarrow N_{total} = \frac{1}{200} + \frac{1}{100} = \frac{3}{200} \approx \frac{1}{67}$$

expected arrivals per day

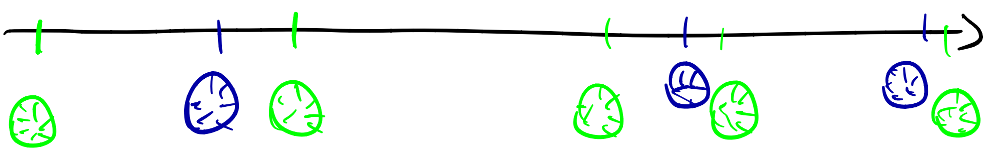
(depending on terminology → new "rate" is 67 days)

b) in the long run, what fraction of replacements are due to failure?

→ compare exercise 2.27 a) $50 = \begin{cases} 30 & \text{female} \\ 20 & \text{male} \end{cases}$

rate $\frac{30}{50} = 3/5$

$$\frac{1}{67} = \begin{cases} \frac{1}{200} & \text{failure} \\ \frac{1}{100} & \end{cases} \Rightarrow \text{share } \frac{\frac{1}{200}}{\frac{1}{67}} \approx \underline{\underline{\frac{1}{3}}}.$$



time

Another way to understand the question:
What is the long term share of blue events out of all events.

2.30

N is a PPP with rate $\lambda \Rightarrow N(t) \sim Poi(\lambda t)$ (*)

N — Calls @rate 4

N_{men} — calls @rate 3

independence

N_{women} — calls @rate 1

(THINNING)

a)

$$\mathbb{P}\left(N_{men}(t=1h)=2,\ N_{women}(t=1h)=3\right)$$

indep.

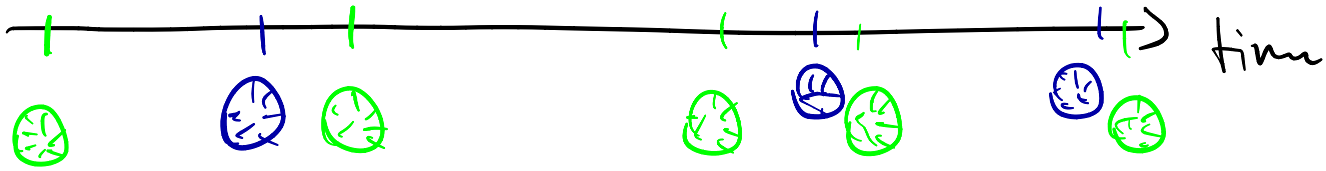
$$= \mathbb{P}\left(N_{men}(t=1h)=2\right)\cdot \mathbb{P}\left(N_{women}(t=1h)=3\right)$$

rate 3

rate 1, one hour

$$\overset{(*)}{=} e^{-3\cdot 1}\frac{(3\cdot 1)^{2}}{2!}\cdot e^{-1\cdot 1}\frac{(1\cdot 1)^{3}}{3!}$$

b) What is the probability that 3 men will have made calls before 3 women have?



$\mathbb{P}$(at least three out of the first five clocks are green)

$= \mathbb{P}$(five clocks out of five are green)

$+ \mathbb{P}$(four — —)

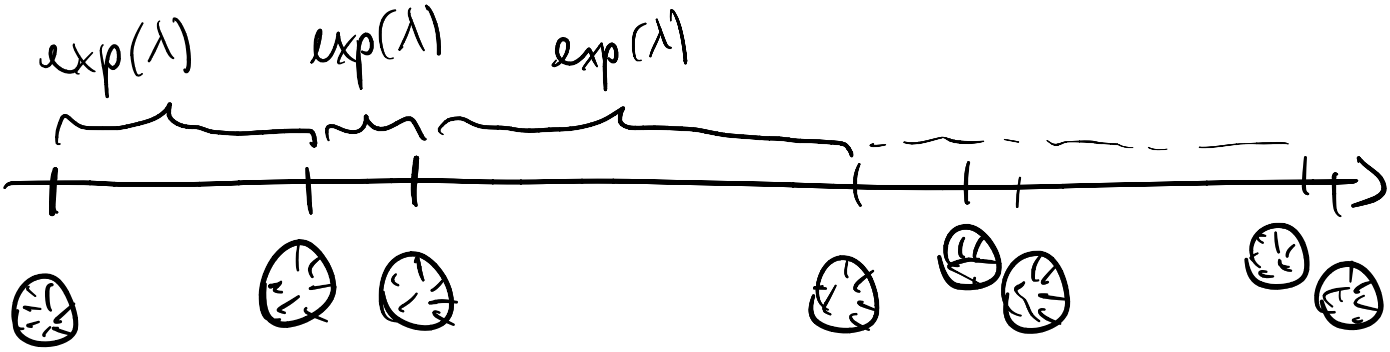
$+ \mathbb{P}$(three — —)

$$= \mathrm{Bin}_{5,3/4}(5) + \mathrm{Bin}_{5,3/4}(4) + \mathrm{Bin}_{5,3/4}(3)$$

$$= \binom{5}{3} \cdot \left(\frac{3}{4}\right)^3 \left(\frac{3}{4}\right)^2 + \binom{5}{4} \left(\frac{3}{4}\right)^4 \left(\frac{3}{4}\right) + \binom{5}{5} \left(\frac{3}{5}\right)^5$$

$= \ldots$

RENEWAL PROCESS



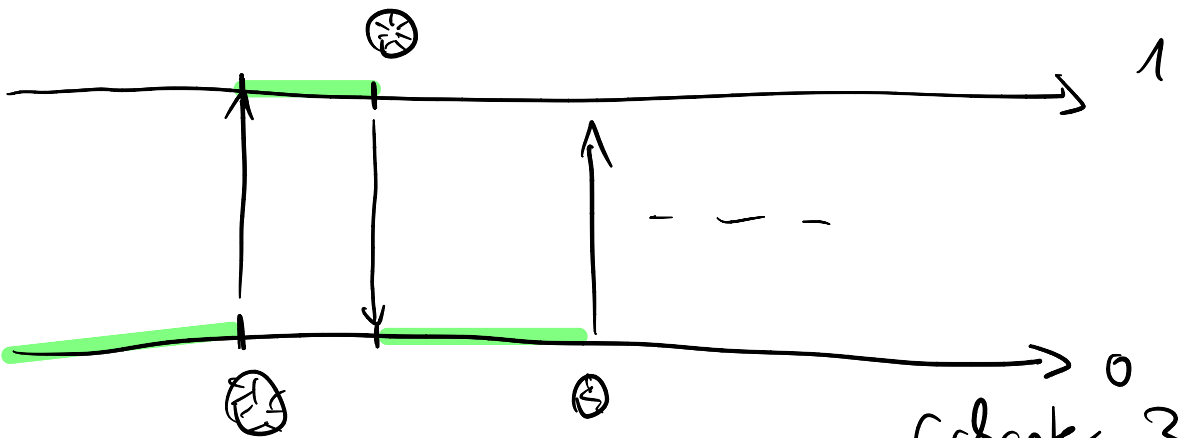
PPP is characterised by exponential waiting times.

Drop (only) this assumption

→ get some "generalised" form of PPP

RENEWAL PROCESS

MOST IMPORTANT APPLICATION:
JUMPING BETWEEN STATES



(chapter 3 of book)

Q: HOW MUCH TIME ON AVERAGE SPENT IN STATE 1 OR 0? → ANSWERS IN THE BOOK