The expected number of visits to a transient state

Assume that we have a Markov chain with a finite state space S and that $j \in S$ is a transient state. We want to study the number of times the Markov chain visits j, i.e.

$$N_j = \#\{n \geq 0 | X_n = j\}$$

(note that we count visits at $n = 0$). Let

$$s_{ij} = E_i[N_j]$$

be the expected value of N_j when we start the process at i. By Lemma 1.9 in the textbook, once the process has reached a recurrent state, it will never go back to a transient state, and hence $s_{ij} = 0$ when i is recurrent. We can therefore disregard all the recurrent states, and if P denotes the transition matrix of the Markov chain, we let P_T denote the *reduced matrix* where we have deleted all the rows and columns belonging to recurrent states. We also let T denote the collection of all transient states.

Observe that if i, j are two *different* transient states, then

$$s_{ij} = \sum_{k \in T} p_{ik} s_{kj},$$

but that for $i = j$, we have

$$s_{ii} = 1 + \sum_{k \in T} p_{ik} s_{ki}$$

as we have to take the first visit into account. If we let S be the matrix with entries s_{ij} where $i, j \in T$, these equations can be written on matrix form as

$$S = I + P_T S$$

where I is the identity matrix. We can rewrite this as

$$(I - P_T)S = I,$$

which tells us that S is invertible and that

$$S = (I - P_T)^{-1}.$$

Hence to find all the expected values s_{ij}, we only have to invert the matrix $I - P_T$.

Example 1: We consider the Markov chain from problem 1, Exam 2005. The state space is $T = \{1, 2, 3, 4, 5\}$ and the transition matrix is

$$P = \begin{pmatrix} 0 & \frac{3}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

It is easy to see that there are two communication classes, $\{1,2\}$ and $\{3,4,5\}$, the first transient and the second recurrent. The reduced matrix is

$$P_T = \begin{pmatrix} 0 & \frac{3}{4} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

and hence

$$S = (I - P_T)^{-1} = \begin{pmatrix} 1 & -\frac{3}{4} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{8}{5} & \frac{9}{5} \\ \frac{4}{5} & \frac{12}{5} \end{pmatrix}.$$

This means that $s_{11} = \frac{8}{5}$ is the expected number of visits to 1 when we start at 1, and $s_{21} = \frac{4}{5}$ is expected number of visits to 1 when we start at 2. Similarly for s_{12} and s_{22}.

We can use the same technique to solve seemingly more complicated problems. Here is an example (from the Mandatory Assignment 2023).

Example 2: A Markov chain has state space $S = \{0,1,2,3,4\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(it's useful to draw a state diagram). We start the process X at state 0 and want to know how many times in average it visits states 0, 1, 2, and 3 before it hits state 4 for the first time.

The trick here is to change 4 into a trap (an absorbing state) before we continue, i.e. we change the transition matrix to

$$\tilde{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Note that the corresponding Markov chain $\tilde{X}$ behaves exactly like X until it hits 4, but when it reaches 4, it is stuck there. For the new Markov chain, state 4 is recurrent while the others are transient. The reduced transition matrix is therefore:

$$\tilde{P}_T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}$$

Using Matlab (or some other program), we compute

$$S = (I - \tilde{P}_T)^{-1} = \begin{pmatrix} \frac{8}{3} & 5 & \frac{8}{3} & 3 \\ \frac{5}{3} & 5 & \frac{8}{3} & 3 \\ \frac{4}{3} & 4 & \frac{10}{3} & 3 \\ 1 & 3 & 2 & 3 \end{pmatrix}$$

According to the theory above, the component s_{ij} is the expected number of times $\tilde{X}$ visits j when it is started in i, i.e. the expected number of times the original process X hits j before it hits 4. As we are starting in $i = 0$, the numbers we are interested in are in the first row of the matrix: Before it reaches state 4, X will in average visit the states 0, 1, 2, 3 a number of $\frac{8}{3}$ times, 5 times, $\frac{8}{3}$ times, and 3 times, respectively.