STK2130: Brownian motion and related processes

by

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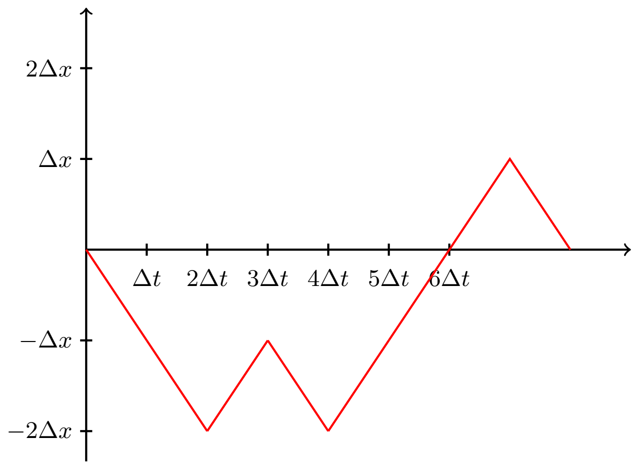
So far we have been studying stochastic processes in either discrete or continuous time. There may seem to be quite a gap between the two kinds of processes, but there is a middle ground: We can study discrete time processes where the time intervals are not 1, but a small number Δt. Often we shall be interested in the limit of these discrete time processes as Δt goes to 0, but this limit procedure is often difficult to describe in precise mathematical terms without using much more advanced mathematical machinery than what we have available. Still, one can often get a good intuitive grasp of the continuous time process by thinking of it as the limit of discrete time processes in a more informal way. This is the philosophy I shall try to follow in the present note.

1 Random walks with vanishing increments

Choose a small positive number Δt, and let

$$\mathbb{T} = \{0, \Delta t, 2\Delta t, 3\Delta t, ...\}$$

be our timeline. We define a stochastic process $X(t)$, $t \in \mathbb{T}$, as follows: $X(0) = 0$, and each time we reach a point s in $\mathbb{T}$, we flip a coin and go a distance Δx up if the result is heads, and a distance Δx down if the result is tails. The figure below shows how the process moves if the coin flips come out tails, tails, heads, tails, heads, heads, heads, tails.



To make the description more mathematical, we let $\{\omega_s\}_{s\in\mathbb{T}}$ be independent random variables which take the values 1 and -1 with probability $\frac{1}{2}$ (think that $\omega_s = 1$ if the coin toss at time s comes out heads, and that $\omega_s = -1$ if the coin toss comes out tails). We then have

$$X(t) = \sum_{s<t} \omega_s \Delta x$$

where the sum is over all $s \in \mathbb{T}$ that is less than t (all later sums will also be over elements in $\mathbb{T}$).

The intuitive idea at the back of our minds is that we want Δt to shrink to 0, but if Δt shrinks to 0, then Δx better shrink to 0, too, as otherwise we shall get a process that jumps around in an uncontrollable manner. The question is how fast should Δx go to 0 compared to Δt? To get a feeling for this, we compute the variance of X. Fix two times $s, t \in \mathbb{T}$, $s < t$. Then

$$E\left[(X(t) - X(s))^2\right] = E\left[\left(\sum_{s \le u < t} \omega_u \Delta x\right)^2\right]$$

$$= E\left[\sum_{s \le u,v < t} \omega_u \omega_v \Delta x^2\right] = \sum_{s \le u,v < t} E\left[\omega_u \omega_v\right] \Delta x^2$$

In the last sum, we see that if $u \neq v$, then ω_u and ω_v are independent, and we get

$$E\left[\omega_u \omega_v\right] = E\left[\omega_u\right] E\left[\omega_v\right] = 0.$$

If $u = v$, we on the other hand have

$$E\left[\omega_u \omega_v\right] = E\left[\omega_u^2\right] = E\left[1\right] = 1.$$

Hence

$$E\left[(X(t) - X(s))^2\right] = \sum_{s \le u < t} \Delta x^2 = \frac{t-s}{\Delta t} \Delta x^2 = (t-s)\frac{\Delta x^2}{\Delta t}$$

as there are $\frac{t-s}{\Delta t}$ elements in $\mathbb{T}$ between s and t. This tells ut that in order for X to keep a reasonable size (neither too small nor too big) as Δt goes to zero, we need to let Δx be of order of magnitude $\sqrt{\Delta t}$. To make things simple, we choose

$$\Delta x = \sqrt{\Delta t}$$

This leads to

$$X(t) = \sum_{s<t} \omega_s \sqrt{\Delta t}$$

and

$$E\left[(X(t) - X(s))^2\right] = t - s.$$

Here are some properties of the random walk X that we would like to generalize later.

Properties of the random walk X

(i) $X(0) = 0$.

(ii) X has independent increments, i.e. if $s_1 < t_1 \le s_2 < t_2 \le \ldots \le s_n < t_n$ are elements of $\mathbb{T}$, then the random variables

$$X(t_1) - X(s_1), X(t_2) - X(s_2), \ldots, X(t_n) - X(s_n)$$

are independent.

(iii) The increments $X(t)-X(s)$ have expectation 0 and variance $t-s$. If $t-s$ is "large" compared to Δt, then the distribution of $X(t)-X(s)$ is close to normal.

The first two of these properties are immediate consequences of the definition of X. It's also clear that the expectation of $X(t)-X(s)$ is zero, and by our calculation above, $\mathrm{Var}[X(t)-X(s)] = E\left[(X(t)-X(s))^2\right] = (t-s)$. Finally, observe that if $t-s$ is large compared to Δt, then $X(t)-X(s) = \sum_{s\leq r<t} \omega_r \sqrt{\Delta t}$ is the sum of a large number of independent, identically distributed random variables, and by the Central Limit Theorem it has a distribution close to normal.

Problems for Section 1

1. Show that $E[X(t)X(s)] = \min\{s,t\}$.
2. In this problem you may want to use that the general solution to the difference equation $x_{n+1}-2x_n+x_{n-1} = K$, where K is a constant, is $x_n = \frac{K}{2}n^2+An+B$, where A and B are constants.
   a) Assume that $T \in \mathbb{T}$. Find the probability that X returns to 0 before it hits the set $\{-T, T\}$.
   b) Show that X is recurrent.
   c) Show that X doesn't have a stationary distribution. Explain why this implies that X is null recurrent.
   d) Let $S_T = \min\{t \in \mathbb{T} : X(t) \in \{-T, T\}\}$. Find $E[S_T]$.

2 Brownian motion

As already mentioned, we would have liked to describe Brownian motion as the limit of the random walks X as Δt goes to 0. Such a construction is possible, but much too complicated for us, and instead we shall use the properties of X as an inspiration for our definition of Brownian motion.

Definition 1 *A* Brownian motion *is a stochastic process* $\{B(t)\}_{t\in[0,\infty)}$ *such that*

(i) $B(0) = 0$.

(ii) B has independent increments, i.e. if $s_1 < t_1 \leq s_2 < t_2 \leq \ldots \leq s_n < t_n$ are elements of $[0,\infty)$, then the random variables

$$B(t_1)-B(s_1), B(t_2)-B(s_2), \ldots, B(t_n)-B(s_n)$$

are independent.

(iii) The increments $B(t)-B(s)$ are normally distributed with mean 0 and variance $t-s$.

Remark. In more advanced texts, there is usually a fourth condition saying that B has continuous paths, i.e. for all $\omega \in \Omega$, the function $t \mapsto B(\omega, t)$ is continuous. This has usually not been required in STK2130, and we drop the condition here as including it would have made some of the old exam problems much harder.

We shall, however, tacitly assume that the paths are continuous in what follows.

A noticeable difference from the processes we have looked at before, is that B takes values in the set of all real numbers $\mathbb{R}$ and not in a finite or countable set S. Hence each $B(t)$ is a *continuous* random variable and has to be treated accordingly.

It may be useful to spell out condition (iii) above in more detail. It tells us that

$$P[B(t) - B(s) \in A] = \int_A \frac{e^{-\frac{x^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}}\,dx \tag{1}$$

for all sets A that are regular enough for the integral to be defined (A may for instance be an interval or a finite or countable union of intervals). If we set $s = 0$, we get

$$P[B(t) \in A] = \int_A \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}}\,dx \tag{2}$$

Formula (2) is excellent for computing where the process is likely to be at a given time t, but often we are more interested in the largest value the process has attained before t. Thanks to the symmetry of Brownian motion, this can be calculated exactly.

Theorem 2 (The Reflction Principle) *Let $X_t = \max_{s\leq t} B_s$ be the largest value of B_s before time t. Then for any $a > 0$,*

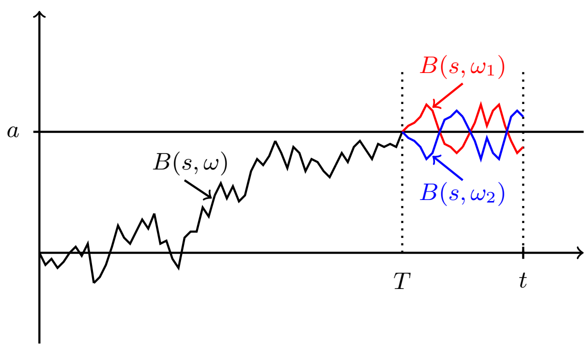
$$P[X_s \geq a] = 2P[B_t \geq a] = \int_a^\infty \frac{\sqrt{2}e^{-\frac{x^2}{2(t-s)}}}{\sqrt{\pi(t-s)}}\,dx.$$

We don't have the machinery to give a complete proof of the reflection property, but here is a rather convincing argument (we are assuming that B has continuous paths):

Define a stopping time T by

$$T = \inf\{s \leq t \mid B(s) \geq a\}$$

if such an s exists, and put $T = t$ otherwise. Fix an ω such that $X_t(\omega) \geq a$. Look at all paths that are equal to "our" path $B(s,\omega)$ up to time T (see figure).



Each such path ω_1 has a mirror image ω_2 whose path is symmetric to the path of ω_1 about the line $y = a$ (see figure). One of these paths will be above a at time t, and the other below a. As the increments of a Brownian motion are symmetric, both events (being below or above a at time t) are equally likely. This means that of all the paths of B such that $X_t \geq a$, one half will be above a at time t and the other half below. Hence

$$P[X_s \geq a] = 2P[B_t \geq a]$$

which is the equality we're looking for.

Assume now that $B_1, B_2, \ldots, B_n$ are n independent Brownian motions. We can then define a process B with values in $\mathbb{R}^n$ by putting

$$B(t) = (B_1(t), B_2(t), \ldots, B_n(t)).$$

This process is called an *n-dimensional Brownian motion.* It has independent increments, and each increment $B(t) - B(s)$ is normally distributed with mean 0 and covariance matrix $(t-s)I_n$, where I_n is the $n \times n$ identity matrix.

You can think of three dimensional Brownian motion as a mathematical model for the irregular motion performed by a small particle suspended in a fluid – a motion caused by collisions with the erratically moving molecules of the fluid. This phenomenon was first observed by the Scottish botanist Robert Brown (1773-1858) who in 1827 used a microscope to study pollen grains suspended in water. Brownian motion is named after him, although the mathematical model we are studying here, was first introduced by the American mathematician Norbert Wiener (1894-1964) in 1923. The term *Wiener process* is sometimes used to distinguish the mathematical model from the physical phenomenon.

Problems for Section 2

1. In this problem B is a Brownian motion.
   a) Show that $-B$ is a Brownian motion.
   b) Fix $s > 0$ and define a new process X by $X(t) = B(t+s)$. Show that X is a Brownian motion.
   c) Let $a > 0$. Define a new process Y by $Y(t) = \sqrt{a}B(\frac{t}{a})$. Show that Y is a Brownian motion.
2. Let B be a Brownian motion. Show that $E[B(t)B(s)] = \min\{s,t\}$.
3. Let B be a Brownian motion. Compute $P[\max_{s \leq 1} B_s \geq 1]$ and give a numerical approximation.
4. Formulate and prove (rigorously) a reflection principle for the random walk X is Section 1.
5. In this problem we shall take a look at a class of processes called *Brownian bridges*. To describe these processes, assume that we are given two numbers $t > 0$ and a. Informally, the Brownian bridge X connecting the origin to the point (t, a) consists of all the paths of a Brownian motion B such that $B(t) = a$. Also, for any $s < t$, the distribution of $X(s)$ should be given by $P[X(s) = x] = P[B(s) = x \mid B(t) = a]$. Unfortunately, the last formula doesn't make sense as we cannot condition on the set $\{B(t) = a\}$ as it has probability 0. We may, however, use Bayes's theorem for densities

   $$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x)f_X(x)}{f_Y(y)}$$

   to find an expression for the density $f_{B(s)|B(t)}(x, a)$. Use this idea to show that $X(s)$ is normally distributed with mean $\frac{sa}{t}$ and variance $s(1 - \frac{s}{t})$.

3 Itô's formula

As already mentioned, Brownian motions are usually supposed to have continuous paths, but in other ways they are extremely irregular; e.g. a typical Brownian path is not differentiable at any point. This leads Brownian paths to behave quite differently from "ordinary" functions in many respects. Itô's formula is one example of such extraordinary behavior, but before we consider it, we should take a look at what to expect from an "ordinary" function.

Assume that $b(s)$ is a continuously differentiable function (think of $b(s)$ as an "ordinary" alternative to a Brownian path), and let f be a differentiable function. By the chain rule

$$\frac{d}{ds} f(b(s)) = f'(b(s))b'(s).$$

If we integrate this expression over the interval $[0, t]$, we get

$$f(b(t)) = f(b(0)) + \int_0^t f'(b(s))b'(s)\, ds$$

It's usual to denote $b'(s)\, ds$ by $db(s)$, i.e.

$$f(b(t)) = f(b(0)) + \int_0^t f'(b(s))\, db(s)$$

Integrals of this kind are usually referred to as a *Stieltjes integrals.*

For Brownian motion, the corresponding formula gets an extra term.

Theorem 3 (Itô's Formula) *If B is a Brownian motion and f is a two times continuously differentiable function, then*

$$f(B(t)) = f(B(0)) + \int_0^t f'(B(s))\, dB(s) + \frac{1}{2}\int_0^t f''(B(s))\, ds$$

In the expression on the right, the first term $\int_0^t f'(B(s))\, dB(s)$ is a so-called Itô-integral, a non-trivial generalization of Stieltjes integrals, and the "extra term" $\frac{1}{2}\int_0^t f''(B(s))\, ds$ is an ordinary Riemann integral.

We cannot prove Itô's formula here (we haven't even defined the Itô-integral, which is quite a task in itself), but we can prove a similar result for the random walk X from Section 1. This will give an indication of where the extra term $\frac{1}{2}\int_0^t f''(B(s))\, ds$ comes from.

We start by observing that if t is an element of the timeline $\mathbb{T}$, we can write

$$f(X(t)) - f(X(0)) = \sum_{s<t} \big[f(X(s+\Delta t)) - f(X(s))\big] \qquad (3)$$

The easiest way to convince oneself that this formula is correct, is perhaps to realize that it just says that the total change in the value of $f(B(s))$ from $s = 0$ to $s = t$, is the sum of all the changes over the subintervals $[s, s + \Delta t]$. One can also see that the formula is correct by writing out the sum on the right and observing that all terms cancel except the "end terms" $f(X(t))$ and $f(X(0))$.

We next apply Taylor's formula with remainder to see that for each s,

$$f(X(s+\Delta t)) - f(X(s)) = f'(X(s))(X(s+\Delta t) - X(s)) + \frac{1}{2} f''(c_s)(X(s+\Delta t) - X(s))^2$$

where c_s is a number between $X(s)$ and $X(s+\Delta t)$. If we introduce the notation

$$\Delta X(s) = X(s+\Delta t) - X(s)$$

and observe that

$$\Delta X(s) = \omega_s \sqrt{\Delta t}$$

and

$$\Delta X(s)^2 = (\omega_s \sqrt{\Delta t})^2 = \Delta t,$$

the formula above can be written more succinctly as

$$f(X(s+\Delta t)) - f(X(s))) = f'(X(s))\Delta X(s) + \frac{1}{2} f''(c_s)\Delta t.$$

If we put this into (3), we get

$$f(X(t)) - f(X(0)) = \sum_{s<t} f'(X(s))\Delta X(s) + \frac{1}{2}\sum_{s<t} f''(c_s)\Delta t.$$

If Δt is small, c_t will be close to $X(t)$, and we get

$$f(X(t)) \approx f(X(0)) + \sum_{s<t} f'(X(s))\Delta X(s) + \frac{1}{2}\sum_{s<t} f''(X_s)\Delta t$$

with better and better approximation the smaller Δt is. This is the discrete version of Itô's formula where the sum $\sum_{s<t} f'(X(s))\Delta X(s)$ replaces the Itô integral $\int_0^t f'(B(s))\, dB(s)$, and the sum $\sum_{s<t} f''(X_s)\Delta t$ replaces the ordinary integral $\int_0^t f''(B(s))\, ds$.

Note that the key to the "extra term" $\int_0^t f''(B(s))\, ds$ in Itô's formula is the observation that since $\Delta X(s)$ is of order of magnitude $\sqrt{\Delta t}$, its square $\Delta X(s)$ is of order of magnitude Δt and must be taken into account.

In the next section, we shall use the following slightly more general version of Itô's formula[1] (the discrete "proof" is the same, just a bit more messy).

Theorem 4 (Itô's Formula, time dependent case) *If B is a Brownian motion and $f(x, y)$ is a function of two variables which is once continuously differentiable in the first variable and twice continuously differentiable in the second variable, then*

$$\begin{aligned} f(t, B(t)) = f(0, B(0)) &+ \int_0^t \frac{\partial f}{\partial y}(s, B(s))\, dB(s) \\ &+ \frac{1}{2}\int_0^t \left(\frac{\partial f}{\partial x}(s, B(s)) + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(s, B(s))\right) ds \end{aligned}$$

Note that the only difference from the first version of Itô's Formula is that we have added a term $\frac{\partial f}{\partial x}(s, B(s))$ to keep track of how changes in the first variable influences the values of $f(s, B(s))$.

[1] There are many versions of Itô's formula of various degrees of generality, and I have chosen two that are easy to formulate in our context.

Problems for Section 3

1. Use Itô's formula to show that

$$\int_0^t B(s)\,dB(s) = \frac{1}{2}(B(t)^2 - t).$$

2. Let g be a differentiable function and apply Theorem 4 to the function $f(x,y) = g(x)y$ to obtain the integration by parts formula

$$\int_0^t g(s)\;dB_s = g(t)B(t) - \int_0^t g'(s)B(s)\;ds$$

for Itô integrals.

4 Geometric Brownian motion

In economics, it's common to use exponential functions

$$f(t) = e^{at}$$

as models of economic growth, but deterministic functions of this kind cannot capture the inherent uncertainty of future developments. A simple model that includes uncertainty is *geometric Brownian motion* which is a stochastic process defined by

$$Y(t) = e^{(a-\frac{b^2}{2})t+bB(t)} \tag{4}$$

where B is a Brownian motion and a, b are real numbers. In finance, geometric Brownian motion is often used to model the value of stocks (this is, for instance, the case in the famous Black-Scholes model). Note that $Y(t)$ is always positive, which is usually a natural assumption when one wants to model the value of something.

The mean value of the geometric Brownian motion X is

$$E[Y(t)] = e^{at},$$

and hence $Y(t)$ fluctuates around the deterministic growth function e^{at}. The size of the fluctuations is governed by the constant b.

The formula (4) for geometric Brownian motion is rather mysterious, especially the $\frac{b^2}{2}$ part. We can use the time dependent version of Itô's formula in Theorem 4 to shed some light on it. Let f be the function

$$f(x,y) = e^{(a-\frac{b^2}{2})x+by}$$

and note that

$$\frac{\partial f}{\partial x} = \left(a - \frac{b^2}{2}\right) e^{(a-\frac{b^2}{2})x+by},$$

$$\frac{\partial f}{\partial y} = b e^{(a-\frac{b^2}{2})x+by},$$

and

$$\frac{\partial^2 f}{\partial y^2} = b^2 e^{(a-\frac{b^2}{2})x+by}.$$

If we stick these expression into the time dependent version of Itô's formula in Theorem 4, we get

$$\begin{aligned}
e^{(a-\frac{b^2}{2})t+bB(t)} &= 1 + \int_0^t b e^{(a-\frac{b^2}{2})t+bB(t)}\,dB(s) \\
&+ \frac{1}{2}\int_0^t \left(\left(a - \frac{b^2}{2}\right) e^{(a-\frac{b^2}{2})t+bB(t)} + \frac{1}{2} b^2 e^{(a-\frac{b^2}{2})t+bB(t)}\right) ds \\
&= 1 + \int_0^t b e^{(a-\frac{b^2}{2})t+bB(t)}\,dB(s) + \int_0^t a e^{(a-\frac{b^2}{2})t+bB(t)}\,ds
\end{aligned}$$

since the two terms with b^2 cancel. Using that $Y(s) = e^{(a-\frac{b^2}{2})s+bB(s)}$ and switching the order of the terms on the right, we get

$$Y(t) = 1 + \int_0^t aY(s)\,ds + \int_0^t bY(s)\,dB(s)$$

This shows that Y satisfies a simple stochastic integral equation. Such equations are usually written on differential form as

$$dY(t) = aY(t)\,dt + bY(t)\,dB(t) \qquad Y(0) = 1 \tag{5}$$

(this equation means exactly the same as the integral equation above, but is easier to read and interpret). If we put b equal to 0 in this expression, we get the usual differential equation for $Y(t) = e^{at}$, and it is natural to think of $Y(t)$ as an exponential function e^{at} whose growth is being disturbed by small random terms $bY(s)\,dB(s)$ of a size proportional to Y.

One of the advantages of the description above, is that the mysterious b^2 term in formula (4) is gone. It is an artifact of the extra term in Itô's formula, and does not enter into the more dynamical description of geometric Brownian motion given by the stochastic differential equation (5).

Problems for Section 4

1. In this problem we shall take a look at a discrete version of geometric Brownian function based on the random walk X from Section 1. If Z is a process with timeline $\mathbb{T} = \{0, \Delta t, 2\Delta t, 3\Delta t, ...\}$, we shall write

   $$\Delta Z(t) = Z(t+\Delta) - Z(t)$$

   for the forward increment of Z at t.

   a) Define a process Y by $Y(0) = 1$ and

   $$\Delta Y(t) = aY(t)\Delta t + bY(t)\Delta X(t)$$

   where a and b are real numbers (this is the discrete version of equation (5)). Show that

   $$Y(t) = \prod_{s<t} \big(1 + a\Delta t + b\Delta X(s)\big)$$

   b) Assume that a and b are chosen such that $1 + a\Delta t - |b|\sqrt{\Delta t} > 0$. Show that

   $$Y(t) = e^{\sum_{s<t} \log\left(1+a\Delta t+b\Delta X(s)\right)}$$

   c) Use a Taylor expansion of $\log(1+x)$ to show that

   $$\begin{aligned} Y(t) &= e^{\sum_{s<t}\left((a-\frac{b^2}{2})\Delta t+b\Delta X(s)+o(\Delta t)\right)} \\ &= e^{(a-\frac{b^2}{2})t+bX(t)+O(\Delta t)}, \end{aligned}$$

   where $O(\Delta t)$ denotes a quantity that is of order of magnitude Δt.