26th February, 2024

# STK2310

## Mandatory assignment 1 of 1

### Submission deadline

Thursday 4th April 2024, 14:30 in Canvas (canvas.uio.no).

### Instructions

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course, and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. You only have one attempt at the assignment, and you need to have it approved in order to take the exam. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

### Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) well before the deadline.

### Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

**Note:** *From the fall semester of 2021, there is a new regime for mandatory assignments. In the new regime, you only have one attempt at each assignment and not two as in earlier years. As the purpose of the new regime is to handle the assignments in a more efficient and pedagogical way and not to fail more students, we are putting more emphasis on* effort *in the grading: As long as you have documented that you have made a serious attempt at the majority of the problems, we will pass you. The best way to document that you have tried, is, of course, to solve the problems, but you can also do it by telling us what you have tried and why it failed. We encourage you to discuss, collaborate, and help each other. Do not hesitate to contact the teachers (preferably well in advance of the deadline) if you have problems.*

You may use software freely to invert matrices, solve systems of equations etc.

**Problem 1.** A Markov chain $X$ has state space $S = \{1, 2, 3, 4, 5\}$ and transition matrix

$$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{3}{4} & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

a) Draw a state diagram for $X$.

b) Find the communication classes and decide which are transient and which are recurrent.

c) Find the period of all the states.

d) Find the stationary distribution $\pi = (\pi_1, \pi_2, \pi_3, \pi_4, \pi_5)$.

e) Show that the distribution of the process does not necessarily converge to the stationary distribution, i.e. show that there are states $i$ and $j$ such that $p_{ij}^{(n)} \nrightarrow \pi_j$ as $n \to \infty$.

f) Assume that $X$ starts in state 1. How many times in average will it visit states 1 and 2?

**Problem 2.** Two teams, A and B, are competing on a TV reality show. Each team has 4 participants, and in each episode of the program the participants are divided into two levels: Four are on the top level and four on the bottom level. At the beginning of the first episode, both teams have two participants on each level. At the beginning of each later episode, the participants are moved between levels according to the following rules. A participant on the top level and a participant on the bottom level are chosen at random (all with the same probability) and the two switch levels. The game continues until one of the teams have all its participants on the top level.

a) Describe the game as a Markov chain. Draw a state diagram and find the transition matrix.

b) What is the expected number of episodes?

c) What is the probability that a team will win if at present it has three participants on the top level?

We modify the rules such that the game continues indefinitely. The only change is that if all the participants on the top level are from the same team, one of them switches levels with a participant from the other team and the game continues as before. In the new version of the game, the teams gather points according to which state they are in: They get 500 points each time they have all their participants on the top level, 100 points each time they have 3 of their participants on the top level, and 0 points each time they have two or less participants on the top level.

d) How many points does a team win per episode in the long run?

**Problem 3.** You're standing at a bus stop waiting for a bus. It's winter in Oslo, and you can just forget about the bus schedule. There are three bus lines you can take, A, B, and C, and you make a mental model where the waiting times for the three bus lines are exponentially distributed with mean 15 minutes, 20 minutes, and 60 minutes, respectively. To make the calculations easier, you count time in hours and express this as $\frac{1}{4}$ hour, $\frac{1}{3}$ hour, and 1 hour.

a) How long do you expect to wait for the first bus to arrive?

b) What is the probability that bus C is the first to arrive?

The first bus arrives, but doesn't stop as it is already full. You now change your model such that each bus that arrives only stops to pick up passengers with probability $\frac{3}{4}$.

c) Show that the waiting time for a bus to stop and pick up passengers is exponentially distributed. What is the rate?

d) Let $N$ be the number of buses that stop to pick up passengers during the first half hour. What is the expectation and standard deviation of $N$ according to the new model? What is the probability that $N \leq 1$, i.e. that at most one bus has stopped during the first half hour?

Good Luck!