UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK2130 — Modelling by Stochastic Processes.

Day of examination: Thursday, August 17th, 2023.

Examination hours: 09.00 – 13.00.

This problem set consists of 2 pages.

Appendices: List of formulas for STK1100 and STK1110.

Permitted aids: Accepted calculator.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All items (Problems 1a, 1b etc.) count equally. If there is a problem you cannot solve, you may still use the result in the sequel. All answers have to be substantiated.

Problem 1 (40 points) A Markov chain X has state space $S = \{0, 1, 2, 3\}$ and transition matrix

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

a) Draw a state diagram which shows how the states communicate.

b) Find the (communication) classes of X and decide which are transient and which are recurrent.

c) Explain that regardless of the starting distribution, the limit probabilities

$$\pi_i = \lim_{n\to\infty} P[X_n = i]$$

exist. Find $\pi_0, \pi_1, \pi_2, \pi_3$.

d) Assume that the process is started in state 0. How many times will it in average visit state 1?

Problem 2 (30 points) A store is open for 10 hours every day. The manager estimates that customers arrive according to a nonhomogeneous Poisson process with intensity function $\lambda(t) = 6t(10 - t)$, $0 \le t \le 10$, where t is the time in hours since the store opened.

a) Sketch the intensity function $\lambda(t)$ and find the mean value function $m(t)$.

b) At what time would the manager expect to have most customers arriving? What is the rate of arrivals at that time?

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c) What is the expected number of customers per day? What is the variance of the numbers of customers in a day?

Problem 3 (40 points) At a hamburger shop, customers are arriving according to a Poisson process with rate 0.4. The shop only serves one customer at a time, and the time it takes to serve a customer is exponentially distributed with mean 2 minutes. Let $X(t)$ be the number of customers waiting to be served at time t (we count customers that have placed their order, but haven't received their food yet, as "waiting to be served").

a) Show that X is a birth and death process. What are the birth rates λ_n and the death rates μ_n?

b) Assume that at time t there are 3 people waiting in line and the first one is being served. The second person says to the third: "I bet a new customer will arrive before I'm getting served." What is the probability that he is right?

According to the general theory of birth and death processes, limit distributions exist if

$$\sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \ldots \lambda_{n-1}}{\mu_1 \mu_2 \ldots \mu_n} < \infty$$

and is then given by

$$P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \ldots \lambda_{n-1}}{\mu_1 \mu_2 \ldots \mu_n}}$$

and

$$P_n = \frac{\lambda_0 \lambda_1 \ldots \lambda_{n-1}}{\mu_1 \mu_2 \ldots \mu_n \left(1 + \sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \ldots \lambda_{n-1}}{\mu_1 \mu_2 \ldots \mu_n}\right)} \qquad \text{for } n > 0.$$

c) Assume that you pass by the hamburger shop long after it has opened. What is the (approximate) probability that there will be no customers waiting to be served? What is the probability that there are exactly two customers in line?

d) What is the expected number of customers waiting to be served?

Problem 4 (20 points) In this problem all random variables are discrete and take values in the set $\mathbb{N} = \{1, 2, 3, \ldots\}$ of natural numbers. Such a random variable X is called *memoryless* if for all $n, k \in \mathbb{N}$, $n > k$, we have

$$P[T > n \mid T > k] = P[T > n - k]$$

(note that this is a slightly different notion of memoryless than we have worked with before).

a) Assume that T is a discrete random variable with

$$P[T = n] = (1 - p)^{n-1} p$$

for all $n \in \mathbb{N}$ (i.e. T is geometrically distributed with probability p). Show that T is memoryless.

b) Assume that S is a discrete, memoryless random variable such that $P[S = 1] = q$. Show that $P[S > n + 1] = (1 - q)P[S > n]$ and use this to show that S is geometrically distributed with probability q.

THE END