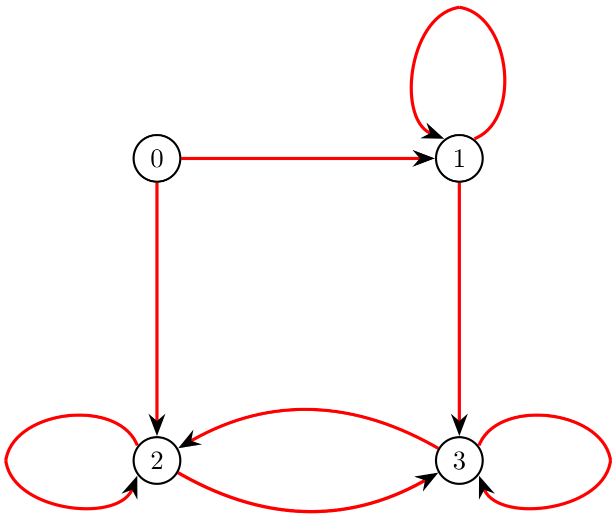
STK2130: Solution to delayed exam, August 17th, 2023

Problem 1 a) The state diagram is:



b) State 0 is its own communication class as it is impossible to get to 0 from the other states. States 2 and 3 communicate, but it is impossible to get from them to 0 and 1, and hence $\{2,3\}$ is a communication class. This means that 1 must also be its own class, and hence the classes are $\{0\}, \{1\}, \{2,3\}$. The classes $\{0\}$ and $\{1\}$ are transient as what they give away to $\{2,3\}$ never comes back. As a Markov chain with finite state space must to have at least one recurrent class, this means that $\{2,3\}$ is recurrent.

c) Note that the states $0,1$ are giving away at least $\frac{1}{3}$ of their content to states $2,3$ at each step without getting anything back. This means that $\pi_0 = \pi_1 = 0$. As all the content ends up in states 2 and 3, we may disregard states 0 and 1, and just consider the Markov chain X' with states 2 and 3 and transition matrix

$$P' = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

As X' is obviously ergodic (i.e., irreducible, recurrent, and aperiodic), it has limit probabilities π_2, π_3 given by $\pi_2 + \pi_3 = 1$ and

$$(\pi_2, \pi_3)P' = (\pi_2, \pi_3)$$

If we write out this equation, we get

$$\frac{1}{4}\pi_2 + \frac{1}{2}\pi_3 = \pi_2$$

and

$$\frac{3}{4}\pi_2 + \frac{1}{2}\pi_3 = \pi_3.$$

Both these equations reduce to $\pi_3 = \frac{2}{3}\pi_2$, and putting this into $\pi_2 + \pi_3 = 1$, we get $\pi_2 + \frac{2}{3}\pi_2 = 1$. Hence $\pi_2 = \frac{2}{5}$ and $\pi_3 = \frac{3}{5}$.

d) If we only consider the transient states 0 and 1, we get the reduced matrix

$$P_T = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & \frac{2}{3} \end{pmatrix}$$

And hence

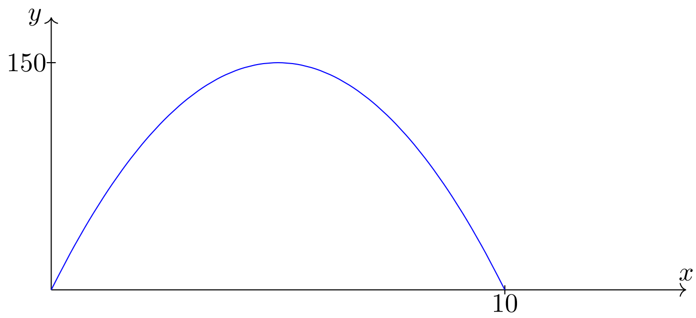
$$I - P_T = \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{3} \end{pmatrix}$$

We are interested in the inverse

$$S = (I - P_T)^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{3} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & \frac{3}{2} \\ 0 & 3 \end{pmatrix}$$

According to the theory, $s_{01} = \frac{3}{2}$ is the average number of times X started in 0 visits 1.

Problem 2 a) The figure shows the intensity function (not in scale):



The mean value function is the integral of the rate function:

$$m(t) = \int_0^t \lambda(s)\,ds = \int_0^t 6s(10-s)\,ds = \int_0^t (60s - 6s^2)\,ds$$

$$= \Big[30s^2 - 2s^3\Big]_0^t = 30t^2 - 2t^3$$

b) As $\lambda'(t) = 60 - 12t$ is 0 for $t = 5$, it is easy to check that the intensity function is maximal for $t = 5$, and that the rate then is $\lambda(5) = 6 \cdot 5 \cdot 5 = 150$. This means that customers arrive at a rate of 150 customers per hour.

c) As $N(t)$ is Poisson distributed with rate $m(t)$ and $m(10) = 30 \cdot 10^2 - 2 \cdot 10^3 = 1000$, we have $E[N(t)] = 1000$ and $\mathrm{Var}(N(t)) = 1000$.

Problem 3 a) A birth corresponds to a new customer joining the queue, and a death corresponds to a customer receiving their hamburger and leaving the queue. As customers arrive at a rate of 0.4, we have $\lambda_n = 0.4$. As serving time is exponential with mean 2 minutes, we get $\mu_n = \frac{1}{2} = 0.5$.

b) The probability that a new customer will arrive before the current customer has been served, is

$$\frac{\lambda_3}{\lambda_3+\mu_3} = \frac{0.4}{0.4+0.5} = \frac{4}{9}.$$

c) As it is long since the shop opened, we may assume that we are close to the limit distribution. Since $\lambda_n = 0.4$ and $\mu_n = 0.5$, the condition

$$\sum_{n=1}^{\infty} \frac{\lambda_0\lambda_1\ldots\lambda_{n-1}}{\mu_1\mu_2\ldots\mu_n} < \infty$$

for a limit distribution to exist, becomes

$$\sum_{n=1}^{\infty} \frac{0.4^n}{0.5^n} = \sum_{n=1}^{\infty} \left(\frac{0.4}{0.5}\right)^n < \infty$$

which is satisfied as $\sum_{n=1}^{\infty} \left(\frac{0.4}{0.5}\right)^n$ is a geometric series with quotient less than one. In fact, by the formula for the sum of a geometric series,

$$\sum_{n=1}^{\infty} \left(\frac{0.4}{0.5}\right)^n = \frac{\frac{0.4}{0.5}}{1-\frac{0.4}{0.5}} = 4.$$

By the formula for the limit distribution, we have

$$P_0 = \frac{1}{1+\sum_{n=1}^{\infty}\frac{\lambda_0\lambda_1\ldots\lambda_{n-1}}{\mu_1\mu_2\ldots\mu_n}} = \frac{1}{1+\sum_{n=1}^{\infty}\left(\frac{0.4}{0.5}\right)^n} = \frac{1}{1+4} = \frac{1}{5}$$

Similarly,

$$P_2 = \frac{\lambda_0\lambda_1}{\mu_1\mu_2\left(1+\sum_{n=1}^{\infty}\frac{\lambda_0\lambda_1\ldots\lambda_{n-1}}{\mu_1\mu_2\ldots\mu_n}\right)} = \frac{0.4^2}{0.5^2\left(1+\sum_{n=1}^{\infty}\left(\frac{0.4}{0.5}\right)^n\right)}$$

$$= \frac{16}{25}\cdot\frac{1}{5} = \frac{16}{125} = 0.128.$$

d) For $n > 0$, we have in general

$$P_n = \frac{\lambda_0\lambda_1\ldots\lambda_{n-1}}{\mu_1\mu_2\ldots\mu_n\left(1+\sum_{n=1}^{\infty}\frac{\lambda_0\lambda_1\ldots\lambda_{n-1}}{\mu_1\mu_2\ldots\mu_n}\right)} = \frac{\left(\frac{4}{5}\right)^n}{5} = \frac{1}{5}\left(\frac{4}{5}\right)^n$$

The expected number of customers waiting is thus

$$E = \sum_{n=1}^{\infty} nP_n = \sum_{n=1}^{\infty} n\frac{1}{5}\left(\frac{4}{5}\right)^n$$

We shall look at two ways to find this sum. The first is to observe that the sum is almost the expectation of a geometric distribution: If we pull a factor $\frac{4}{5}$ out of the sum, we get

$$E = \frac{4}{5}\sum_{n=1}^{\infty} n\frac{1}{5}\left(\frac{4}{5}\right)^{n-1},$$

where the new sum $\sum_{n=1}^{\infty} n\frac{1}{5}\left(\frac{4}{5}\right)^{n-1}$ is the expectation of a geometric random variable Y with probability $p = \frac{1}{5}$. Since $E[Y] = \frac{1}{p} = 5$ (see formula sheet), we get

$$E = \frac{4}{5} \cdot 5 = 4.$$

For the second way to find the sum, we first observe that by summing a geometric series, we get

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

for all x with $|x| < 1$. Differentiating both sides, we get

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1},$$

and hence

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$$

With $x = \frac{4}{5}$, we get that the expected number of customers in the queue is

$$\frac{1}{5}\sum_{n=1}^{\infty} n\left(\frac{4}{5}\right)^n = \frac{1}{5}\frac{\frac{4}{5}}{(1-\frac{4}{5})^2} = 4$$

Problem 4 a) As a preliminary calculation, we observe that by the sum formula for a geometric series

$$P[T > m] = \sum_{n=m+1}^{\infty} P[T = n] = \sum_{n=m+1}^{\infty} (1-p)^{n-1}p = p\frac{(1-p)^m}{1-(1-p)} = (1-p)^m.$$

To solve the problem, we need to compute

$$P[T > n \mid T > k] = \frac{P([T > n] \cap [T > k])}{P[T > k]}$$

As $[T > n] \cap [T > k] = [T > n]$ (if T is larger than n, it is automatically larger than k since k is less than n), we get

$$P[T > n \mid T > k] = \frac{P([T > n])}{P[T > k]} = \frac{(1-p)^n}{1-p)^k} = (1-p)^{n-k} = P[T > n-k]$$

b) Since S is memoryless, we have

$$P[S > n+1 \mid S > n] = P[S > 1] = 1-q$$

On the other hand,

$$P[S > n+1 \mid S > n] = \frac{P([S > n+1] \cap [S > n])}{P[S > n]} = \frac{P[S > n+1]}{P[S > n]}$$

Combining the two expressions above, we get $P[S > n+1] = (1-q)P[S > n]$. As $P[S > 1] = 1-q$, it follows by induction that $P[S > n] = (1-q)^n$. Hence

$$P[S = n] = P[S > n-1] - P[S > n] = (1-q)^{n-1} - (1-q)^n$$
$$= (1-q)^{n-1}((1-(1-q)) = (1-q)^{n-1}q$$

which shows that S is geometric with probability q.