UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK2130 — Modelling by Stochastic Processes.

Day of examination: Thursday, June 5th, 2024.

Examination hours: 15.00 – 19.00.

This problem set consists of 3 pages.

Appendices: List of formulas for STK1100 and STK1110.

Permitted aids: Accepted calculator.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All items (Problems 1a, 1b etc.) count equally. If there is a problem you cannot solve, you may still use the result in the sequel. All answers have to be substantiated. At the end of the problem set there are some formulas you may find useful along the way.

Problem 1 (60 points) A Markov chain X has state space $S = \{1, 2, 3, 4, 5\}$ and transition matrix

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{2}{3} & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

a) Draw a state diagram which shows how the states communicate.

b) Find the communication classes of X and decide which ones are transient and which are recurrent.

c) Find the stationary distribution $\pi = (\pi_1, \pi_2, \pi_3, \pi_4, \pi_5)$.

d) Does the Markov chain always converge to the stationary distribution, i.e., is $\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j$ for all $i, j \in S$?

e) Assume that X starts in state 1. How long does it take for it to hit the set $\{4, 5\}$? Depending on how you solve the problem, you may or may not need that

$$\begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{4} & 1 & -\frac{1}{2} \\ -\frac{2}{3} & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{8}{3} & \frac{4}{3} & 2 \\ \frac{14}{9} & \frac{16}{9} & \frac{5}{3} \\ \frac{16}{9} & \frac{8}{9} & \frac{7}{3} \end{pmatrix}$$

f) If X starts in state 1, what is the probability that it hits state 4 before it hits state 5?

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Problem 2 (30 points) The service time of a service counter is exponentially distributed with mean 3 minutes. New customers arrive at the counter according to a Poisson process N_1 with rate $\lambda_1 = \frac{1}{5}$.

a) Let $X_1(t)$ be the number of customers waiting in line to be served at time t (including the one who is currently being served). Find the stationary distribution π for X_1. What proportion of time is the counter idle in the long run?

When customers are finished in the queue above, they will with probability $\frac{2}{3}$ leave the system, and with probability $\frac{1}{3}$ be sent to another queue. The service time in this queue is exponential with mean 2 minutes. In addition to the customers coming from the first queue, new customers are arriving at this queue according to an independent Poisson process with rate $\frac{1}{3}$. After the customers are finished at this counter, they leave the system.

b) Assume that X_1 is started with the stationary distribution. Let $N_2(t)$ be the total number of customers that have entered the second queue up to time t. What kind of process is N_2? Be as specific as possible.

c) As above, let $X_1(t)$ be the number of customers waiting in line for the first counter at time t, and let $X_2(t)$ be the number of customers waiting in line for the second counter (in both cases we include the customer who is currently being served). Find the stationary distribution $\tilde{\pi}(n, m)$ for the process $X = (X_1, X_2)$. In the long run, how much of the time will both counters be idle simultaneously?

Problem 3 (50 points) Fredrik likes to go fishing, but even more he likes to make mathematical models for what he might catch. Fredrik is fishing two kinds of fish, whiting and cod. In his latest model, the number of whitings he catches is a Poisson process with rate 4 (time is counted in hours). The number of cods also follow a Poisson process, but with a mean of one cod per hour. We assume that the two processes (the one for whiting and the one for cod) are independent.

a) How long does Fredrik in average have to wait for his first catch? What is the probability that it is a whiting?

b) What is the probability that the first two catches are whitings?

c) Assume that X, Y_1, Y_2 are independent random variables. Assume also that X equals 1 with probability p and 0 with probability $1 - p$. Let $Z = XY_1 + (1 - X)Y_2$. Show that $E(Z) = pE(Y_1) + (1 - p)E(Y_2)$ and that $E(Z^2) = pE(Y_1^2) + (1 - p)E(Y_2^2)$.

d) The whitings Fredrik catches have an average weight of 1 kg with a standard deviation of $\frac{1}{2}$ kg. The cods he catches, have an average weight of 3 kg with a standard deviation of $\frac{3}{2}$ kg. Find the mean and the standard deviation of the first catch.

e) Let $Y(t)$ be the total weight of the fish Fredrik catches during the first t hours. Explain that Y is a compound Poisson process, and find the mean and standard deviation of $Y(t)$.

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Formulas from the syllabus that you may find useful

Stationary distributions for birth and death processes:

$$\pi_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \dots \lambda_{n-1}}{\mu_1 \mu_2 \dots \mu_n}}$$

and

$$\pi_n = \frac{\lambda_0 \lambda_1 \dots \lambda_{n-1}}{\mu_1 \mu_2 \dots \mu_n \left(1 + \sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \dots \lambda_{n-1}}{\mu_1 \mu_2 \dots \mu_n}\right)} \quad \text{for } n > 0$$

provided the sums converge.

Random sums: Assume that $Z_1, Z_2, Z_3, \dots$ are i.i.d random variables with finite second moments and that N is a Poisson random variable independent of the Z_n's. Let $Y = \sum_{n=1}^{N} Z_n$. Then the mean and variance of Y is given by:

$$E[Y] = E[N]E[Z_n]$$

$$\text{var}[Y] = E[N]E[Z_n^2]$$

THE END