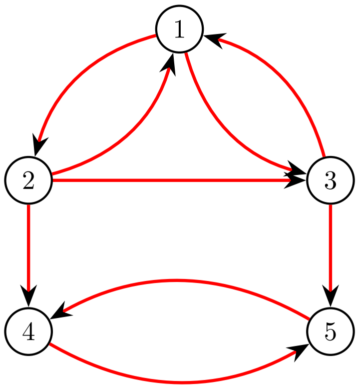
STK2130: Solution to the exam spring 2024

Problem 1 a) The state diagram:



b) From the diagram we see that it is possible to get from all of the states $1, 2, 3$ to any of the other states by following the arrows, but that it is impossible to get from states 4 and 5 to states 1, 2, and 3. This means that $\{1, 2, 3\}$ is a communication class, and since 4 and 5 communicate, $\{4, 5\}$ is also a communication class. The class $\{1, 2, 3\}$ is transient as it is losing mass to $\{4, 5\}$ that it doesn't get back. As a Markov chain with a finite state space always has a recurrent class, this means that $\{4, 5\}$ is recurrent.

c) Since the states 1, 2, and 3 are transient, we know that $\pi_1 = \pi_2 = \pi_3 = 0$. The equation $\pi = \pi P$ then becomes

$$(0, 0, 0, \pi_4, \pi_5) = (0, 0, 0, \pi_4, \pi_5) \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{2}{3} & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} = (0, 0, 0, \pi_5, \pi_4),$$

and hence we need to have $\pi_4 = \pi_5$. As we also need to have $\pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 = 1$, we must have $\pi_4 = \pi_5 = \frac{1}{2}$. Hence the stationary distribution is $\pi = (0, 0, 0, \frac{1}{2}, \frac{1}{2})$.

d) Observe that the states 4 and 5 have period 2 as we can only go back and forth between them. This means that

$$p_{4,4}^{(n)} = \begin{cases} 1 & \text{if } n \text{ is even} \\ \\ 0 & \text{if } n \text{ is odd} \end{cases},$$

and hence $\lim_{n\to\infty} p_{4,4}^{(n)}$ does not exist, and the Markov chain does not converge to the stationary distribution.

e) We give two solutions of this problem:

Solution 1: If we start with the transition matrix P and delete the rows and columns associated with states 4 and 5, we get the reduced matrix

$$r = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & 0 & \frac{1}{2} \\ \frac{2}{3} & 0 & 0 \end{pmatrix}$$

Obviously, $I - r = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{4} & 1 & -\frac{1}{2} \\ -\frac{2}{3} & 0 & 1 \end{pmatrix}$, and by the formula in the problem

$$(I-r)^{-1} = \begin{pmatrix} \frac{8}{3} & \frac{4}{3} & 2 \\ \frac{14}{9} & \frac{16}{9} & \frac{5}{3} \\ \frac{16}{9} & \frac{8}{9} & \frac{7}{3} \end{pmatrix}$$

Now

$$(I-r)^{-1}\mathbf{1} = \begin{pmatrix} \frac{8}{3} & \frac{4}{3} & 2 \\ \frac{14}{9} & \frac{16}{9} & \frac{5}{3} \\ \frac{16}{9} & \frac{8}{9} & \frac{7}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ 5 \end{pmatrix},$$

and according to the theory in the textbook, 6 is the average number of steps it takes to get from 1 to $\{4,5\}$.

Solution 2: Let x, y, z be the average number of steps it takes to get to $\{4,5\}$ from 1, 2, and 3, respectively. By "one step analysis",

$$\begin{aligned} x &= 1 + \frac{1}{2}y + \frac{1}{2}z \\ y &= 1 + \frac{1}{4}x + \frac{1}{2}z + \frac{1}{4}\cdot 0 \\ z &= 1 + \frac{2}{3}x + \frac{1}{3}\cdot 0 \end{aligned}$$

Rearranging the equations, we get

$$\begin{aligned} x - \frac{1}{2}y - \frac{1}{2}z &= 1 \\ -\frac{1}{4}x + y - \frac{1}{2}z &= 1 \\ -\frac{2}{3}x + z &= 1 \end{aligned}$$

If we solve this system either from scratch or by using the inverse matrix in the problem set, we get $x = 6$, $y = 5$, $z = 5$. Hence the average time to get from 1 to $\{4,5\}$ is 6.

f) Let x be the probability that X started in 1 hits 4 before 5, and let y, z be the corresponding probabilities for X started in 2 and 3, respectively. By "one step analysis",

$$\begin{aligned} x &= \frac{1}{2}y + \frac{1}{2}z \\ y &= \frac{1}{4}x + \frac{1}{2}z + \frac{1}{4}\cdot 1 \\ z &= \frac{2}{3}x + \frac{1}{3}\cdot 0 \end{aligned}$$

Rearranging the equations, we get

$$x - \frac{1}{2}y - \frac{1}{2}z = 0$$
$$-\frac{1}{4}x + y - \frac{1}{2}z = \frac{1}{4}$$
$$-\frac{2}{3}x + z = 0$$

Solving these equations (again it is possible to use the inverse matrix from e)), we get $x = \frac{1}{3}$, $y = \frac{4}{9}$, $z = \frac{2}{9}$. Hence the probability of hitting 4 before 5 when we start from 1 is $\frac{1}{3}$.

Problem 2 a) The length of the queue follows a birth and death process with rates $\lambda_n = \frac{1}{5}$ and $\mu_n = \frac{1}{3}$. Using the formulas on the problem sheet plus that the sum of the geometric series $\sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n$ is $\frac{5}{2}$, we see that the stationary distribution is given by

$$\pi_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \frac{\lambda_0\lambda_1\ldots\lambda_{n-1}}{\mu_1\mu_2\ldots\mu_n}} = \frac{1}{1 + \sum_{n=1}^{\infty} \frac{\left(\frac{1}{5}\right)^n}{\left(\frac{1}{3}\right)^n}} = \frac{1}{\sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n} = \frac{2}{5}$$

and for $n > 0$

$$\pi_n = \frac{\lambda_0\lambda_1\ldots\lambda_{n-1}}{\mu_1\mu_2\ldots\mu_n\left(1 + \sum_{n=1}^{\infty} \frac{\lambda_0\lambda_1\ldots\lambda_{n-1}}{\mu_1\mu_2\ldots\mu_n}\right)} = \frac{\left(\frac{1}{5}\right)^n}{\left(\frac{1}{3}\right)^n \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n} = \frac{2}{5}\left(\frac{3}{5}\right)^n$$

Note that this last formula also works for $n = 0$, and hence

$$\pi_n = \frac{2}{5}\left(\frac{3}{5}\right)^n$$

for all n.

In the long run, the proportion of time the counter is idle equals $\pi_0 = \frac{2}{5}$.

b) According to the theory, the customers are leaving the first queue according to a Poisson process with rate $\frac{1}{5}$. The fraction that continues to the other counter, does it according to a Poisson process with rate $p\lambda_1 = \frac{1}{3} \cdot \frac{1}{5} = \frac{1}{15}$ (thinning). Hence counter number two is fed by two independent Poisson processes with rates $\frac{1}{15}$ and $\frac{1}{3}$, respectively. The sum $N_2(t)$ is a Poisson process with rate $\frac{1}{15} + \frac{1}{3} = \frac{2}{5}$.

c) When the first counter is in equilibrium, the second counter receives new customers at a rate of $\frac{2}{5}$ and serves them at a rate of $\frac{1}{2}$, hence the queue behaves like a birth and death process with $\lambda_n = \frac{2}{5}$ and $\mu_n = \frac{1}{2}$. Using the same formulas as above, we see that the stationary state π' is given by

$$\pi'_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \frac{\lambda_0\lambda_1\ldots\lambda_{n-1}}{\mu_1\mu_2\ldots\mu_n}} = \frac{1}{\sum_{n=0}^{\infty} \frac{\left(\frac{2}{5}\right)^n}{\left(\frac{1}{2}\right)^n}}$$

and

$$\pi'_n = \frac{\lambda_0\lambda_1\ldots\lambda_{n-1}}{\mu_1\mu_2\ldots\mu_n\left(1 + \sum_{n=1}^{\infty} \frac{\lambda_0\lambda_1\ldots\lambda_{n-1}}{\mu_1\mu_2\ldots\mu_n}\right)} = \frac{\left(\frac{2}{5}\right)^n}{\left(\frac{1}{2}\right)^n \sum_{n=0}^{\infty} \frac{\left(\frac{2}{5}\right)^n}{\left(\frac{1}{2}\right)^n}}$$

for $n > 0$. Summing a geometric series, we see that

$$\sum_{n=0}^{\infty} \frac{\left(\frac{2}{5}\right)^n}{\left(\frac{1}{2}\right)^n} = \sum_{n=0}^{\infty} \left(\frac{4}{5}\right)^n = 5$$

and the formulas above simplify to

$$\pi'_0 = \frac{1}{5}$$

and

$$\pi'_n = \frac{4^n}{5^{n+1}}$$

for $n > 0$. Again, the last formula also works for $n = 0$, and we get

$$\pi'_n = \frac{4^n}{5^{n+1}}$$

for all n.

This means that the combined stationary state for both queues is

$$\hat{\pi}(n,m) = \pi_n \pi'_m = \frac{2}{5}\left(\frac{3}{5}\right)^n \cdot \frac{4^m}{5^{m+1}} = 2\frac{3^n 4^m}{5^{n+m+2}}$$

The counters will be idle simultaneously a proportion of time that equals $\hat{\pi}(0,0) = \frac{2}{25}$, i.e. 8 % of the time.

Problem 3 a) The waiting time T_w for the first whiting is exponential with rate $\lambda_w = 4$, and the waiting time T_c for the first cod is exponential with rate $\lambda_c = 1$. The waiting time for the first catch (the minimum of T_w and T_c) is then $\lambda = \lambda_w + \lambda_c = 4 + 1 = 5$. This means that in average Fredrik has to wait $\frac{1}{\lambda} = \frac{1}{5}$ hour for the first catch, i.e. 12 minutes.

The probability that the first catch is a whiting, is $\frac{\lambda_w}{\lambda_w + \lambda_c} = \frac{4}{5}$.

b) According to a), the probability of the first catch being a whiting is $\frac{4}{5}$. When the first fish has been caught, the process starts anew as exponential distributions are memoryless, and hence the probability of the second catch being a whiting is also $\frac{4}{5}$. This means that the probability of the first two catches being whitings is $\frac{4}{5} \cdot \frac{4}{5} = \frac{16}{25}$.

c) Using the independence, we have

$$E[Z] = E[XY_1 + (1-X)Y_2] = E[X]E[Y_1] + E[1-X]E[Y_2] = pE[Y_1] + (1-p)E[Y_2]$$

Using the independence and observing that $X(1-X) = 0$, we get

$$E[Z^2] = E[(XY_1 + (1-X)Y_2)^2] = E[X^2Y_1^2] + 2E[X(1-X)Y_1Y_2] + E[(1-X)^2Y_2^2]$$

$$= E[X^2]E[Y_1^2] + 0 + E[(1-X)^2]E[Y_2^2] = pE[Y_1^2] + (1-p)E[Y_2^2].$$

d) The first (and any later) catch is a whiting with probability $\frac{4}{5}$ and a cod with probability $\frac{1}{5}$. If Y_1 is the distribution of the weight of the whitings, and Y_2 is the distribution of the weight of the cods, the weight of the first catch (and any later catch) has the distribution of $Z = XY_1 + (1-X)Y_2$ where X is independent of Y_1, Y_2, and $X = 1$ with probability $\frac{4}{5}$ and 0 with probability $\frac{1}{5}$. Applying the result from c), we see that

$$E[Z] = \frac{4}{5}E[Y_1] + \frac{1}{5}E[Y_2] = \frac{4}{5} \cdot 1 + \frac{1}{5} \cdot 3 = \frac{7}{5}$$

(you can, of course, also find this much less formally!).

Since $E[Y_1^2] = \text{var}[Y_1] + E[Y_1]^2 = (\frac{1}{2})^2 + 1 = \frac{5}{4}$ and $E[Y_2^2] = \text{var}[Y_2] + E[Y_2]^2 = (\frac{3}{2})^2 + 9 = \frac{45}{4}$, we get from c) that

$$E[Z^2] = \frac{4}{5} \cdot \frac{5}{4} + \frac{1}{5} \cdot \frac{45}{4} = \frac{13}{4}.$$

Hence

$$\text{var}[Z] = E[Z^2] - E[Z]^2 = \frac{13}{4} - \left(\frac{7}{5}\right)^2 = \frac{13}{4} - \frac{49}{25} = \frac{129}{100}$$

and the standard deviation is

$$\sigma_Z = \sqrt{\frac{129}{100}} = \frac{\sqrt{129}}{10} \approx 1.136.$$

e) The weight of any catch has the distribution of the random variable Z in part d). Hence if we have caught $N(t)$ fish by time t, the weight of the catch at time t is

$$Y(t) = \sum_{n=1}^{N(t)} Z_n$$

where the Z_n's are independent copies of Z. This is a compound Poisson process, and by the first formula for random sums on the problem sheet,

$$E[Y(t)] = E[N(t)]E[Z] = (5t) \cdot \frac{7}{5} = 7t$$

since N is a Poisson process of rate 5. Similarly, by the second formula on the formula sheet,

$$\text{var}[Y(t)] = E[N(t)]E[Z^2] = (5t) \cdot \frac{13}{4} = \frac{65}{4}t.$$