STK3405 – Week 35

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Section 2.5

***k*-out-of-*n* systems**

k-out-of-n systems

A k-out-of-n system is a binary monotone system (C, ϕ) where $C = \{1, \ldots, n\}$ which functions if and only if at least k out of the n components are functioning.

Let the component state variable of component i be X_i, $i \in C$, and let the vector of component state variables be $\boldsymbol{X} = (X_1, \ldots, X_n)$.

The structure function, ϕ, can then be written:

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} X_i \geq k \\ 0 & \text{otherwise.} \end{cases}$$

An *n*-out-of-*n* system = A series system

An *n*-out-of-*n* system is the same as a series system:

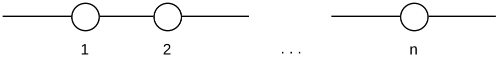


Figure: A reliability block diagram of an *n*-out-of-*n* system.

A 1-out-of-n system = A parallel system

A 1-out-of-n system is the same as a parallel system:

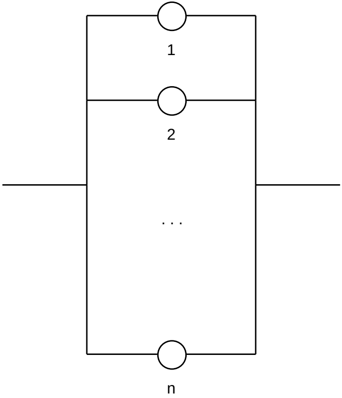


Figure: A reliability block diagram of an 1-out-of-n system.

A 2-out-of-3 system

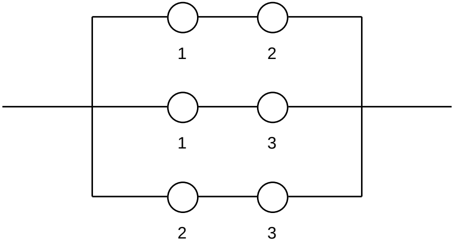


Figure: A reliability block diagram of a 2-out-of-3 system.

For a 2-out-of-3 system to function 2 out of 3 components must function. There are 3 possible subsets of components which contains 2 components: $\{1,2\}$, $\{1,3\}$, $\{2,3\}$.

A 2-out-of-3 system

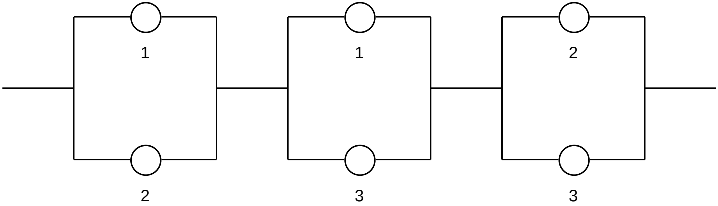


Figure: A reliability block diagram of a 2-out-of-3 system.

For a 2-out-of-3 system to fail 2 out of 3 components must fail. There are 3 possible subsets of components which contains 2 components: $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$.

A 3-out-of-4 system

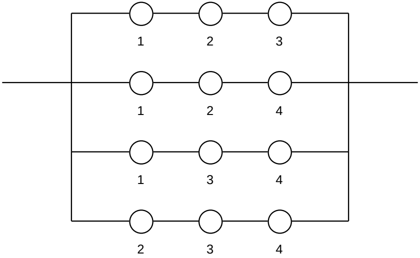


Figure: A reliability block diagram of a 3-out-of-4 system.

For a 3-out-of-4 system to function 3 out of 4 components must function. There are 4 possible subsets of components which contains 3 components: $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{2, 3, 4\}$.

A 2-out-of-4 system

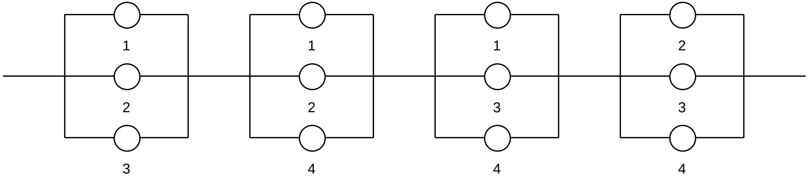


Figure: A reliability block diagram of a 2-out-of-4 system.

For a 2-out-of-4 system to fail 3 out of 4 components must fail. There are 4 possible subsets of components which contains 3 components: $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$, $\{2,3,4\}$.

The reliability of a k-out-of-n system

In order to evaluate the reliability of a k-out-of-n system it is convenient to introduce the following random variable:

$$S = \sum_{i=1}^{n} X_i.$$

Thus, S is the number of functioning components. This implies that:

$$h = P(\phi(\boldsymbol{X}) = 1) = P(S \geq k).$$

The reliability of a k-out-of-n system (cont.)

If the component states are *independent*, and the component reliabilities are all *equal*, i.e., $p_1 = \cdots = p_n = p$, the random variable S is a binomially distributed random variable, and we have:

$$P(S = i) = \binom{n}{i} p^i (1 - p)^{n-i}.$$

Hence, the reliability of the system is given by:

$$h(\boldsymbol{p}) = h(p) = P(S \geq k) = \sum_{i=k}^{n} \binom{n}{i} p^i (1 - p)^{n-i}$$

The reliability of a 2-out-of-3 system

EXAMPLE: Let (C, ϕ) be a 2-out-of-3 system where the component states are independent, and where $p_1 = p_2 = p_3 = p$. We then have:

$$P(S = 2) = \binom{3}{2} p^2 (1-p)^1 = 3p^2(1-p)$$

$$P(S = 3) = \binom{3}{3} p^3 (1-p)^0 = p^3.$$

Hence, the reliability of the system is:

$$h = P(S \geq 2) = 3p^2(1-p) + p^3 = 3p^2 - 2p^3.$$

The reliability of a 3-out-of-4 system

EXAMPLE: Let (C, ϕ) be a 3-out-of-4 system where the component states are independent, and where $p_1 = p_2 = p_3 = p_4 = p$. We then have:

$$P(S = 3) = \binom{4}{3} p^3 (1-p)^1 = 4p^3(1-p)$$

$$P(S = 4) = \binom{4}{4} p^4 (1-p)^0 = p^4.$$

Hence, the reliability of the system is:

$$h = P(S \geq 3) = 4p^3(1-p) + p^4 = 4p^3 - 3p^4.$$

The reliability of a k-out-of-n system (cont.)

When the component reliabilities are unequal, explicit analytical expressions for the distribution of S is not so easy to derive.

Let S be a stochastic variable with values in $\{0, 1, \ldots, n\}$. We then define the *generating function* of S as:

$$G_S(y) = E[y^S] = \sum_{s=0}^{n} y^s P(S = s).$$

When a random variable S is the sum of a set of independent random variables $X_1, \ldots, X_n$, the generating function of S is the product of the generating functions of $X_1, \ldots, X_n$. By using this property it is possible to construct a very efficient algorithm for calculating the distribution of S.

We will return to this issue in an exercise.

The reliability of a 2-out-of-3 system

EXAMPLE: Let (C, ϕ) be a 2-out-of-3 system where the component states are independent with reliabilities p_1, p_2, p_3, We then have:

$$P(S = 2) = p_1 p_2 (1 - p_3) + p_1 (1 - p_2) p_3 + (1 - p_1) p_2 p_3$$

$$P(S = 3) = p_1 p_2 p_3$$

Hence, the reliability of the system is:

$$\begin{aligned} h &= p_1 p_2 (1 - p_3) + p_1 (1 - p_2) p_3 + (1 - p_1) p_2 p_3 + p_1 p_2 p_3 \\ &= p_1 p_2 + p_1 p_3 + p_2 p_3 - 2 p_1 p_2 p_3. \end{aligned}$$

The reliability of a 3-out-of-4 system

EXAMPLE: Let (C, ϕ) be a 3-out-of-4 system where the component states are independent with reliabilities p_1, p_2, p_3, p_4, We then have:

$$
\begin{aligned}
P(S = 3) &= p_1 p_2 p_3 (1 - p_4) + p_1 p_2 (1 - p_3) p_4 \\
&\quad + p_1 (1 - p_2) p_3 p_4 + (1 - p_1) p_2 p_3 p_4 \\
P(S = 4) &= p_1 p_2 p_3 p_4
\end{aligned}
$$

Hence, the reliability of the system is:

$$
\begin{aligned}
h &= p_1 p_2 p_3 (1 - p_4) + p_1 p_2 (1 - p_3) p_4 \\
&\quad + p_1 (1 - p_2) p_3 p_4 + (1 - p_1) p_2 p_3 p_4 \\
&\quad + p_1 p_2 p_3 p_4 \\
&= p_1 p_2 p_3 + p_1 p_2 p_4 + p_1 p_3 p_4 + p_2 p_3 p_4 - 3 p_1 p_2 p_3 p_4
\end{aligned}
$$

Chapter 3

Basic reliability calculation methods

Pivotal decompositions

Pivotal decompositions

Theorem

Let (C, ϕ) be a binary monotone system. We then have:

$$\phi(\boldsymbol{x}) = x_i \phi(1_i, \boldsymbol{x}) + (1 - x_i)\phi(0_i, \boldsymbol{x}), \quad i \in C. \tag{1}$$

Similarly, for the reliability function of a binary monotone system where the component state variables are independent, we have

$$h(\boldsymbol{p}) = p_i h(1_i, \boldsymbol{p}) + (1 - p_i) h(0_i, \boldsymbol{p}), \quad i \in C. \tag{2}$$

Pivotal decompositions (cont.)

PROOF: Let $i \in C$, and consider two cases:

CASE 1. $x_i = 1$. Then the right-hand side of (1) becomes:

$$\phi(1_i, \boldsymbol{x}).$$

Hence, $\phi(\boldsymbol{x}) = \phi(1_i, \boldsymbol{x})$, so (1) holds in this case.

CASE 2. $x_i = 0$. Then the right-hand side of (1) becomes:

$$\phi(0_i, \boldsymbol{x}),$$

Hence, $\phi(\boldsymbol{x}) = \phi(0_i, \boldsymbol{x})$, so (1) holds in this case as well.

Equation (2) is proved by replacing the vector $\boldsymbol{x}$ by $\boldsymbol{X}$ in (1), and taking the expectation.

Pivotal decompositions (cont.)

Corollary

Let (C, ϕ) be a binary monotone system. We then have:

$$\begin{aligned}\phi(\boldsymbol{x}) &= x_i x_j \phi(1_i, 1_j, \boldsymbol{x}) + x_i(1 - x_j)\phi(1_i, 0_j, \boldsymbol{x}) \\ &\quad + (1 - x_i)x_j\phi(0_i, 1_j, \boldsymbol{x}) + (1 - x_i)(1 - x_j)\phi(0_i, 0_j, \boldsymbol{x}), \quad i, j \in C.\end{aligned}$$

Similarly, for the reliability function of a binary monotone system where the component state variables are independent, we have:

$$\begin{aligned}h(\boldsymbol{p}) &= p_i p_j h(1_i, 1_j, \boldsymbol{p}) + p_i(1 - p_j)h(1_i, 0_j, \boldsymbol{p}) \\ &\quad + (1 - p_i)p_j h(0_i, 1_j, \boldsymbol{p}) + (1 - p_i)(1 - p_j)h(0_i, 0_j, \boldsymbol{p}), \quad i, j \in C.\end{aligned}$$

PROOF: Use the pivotal decomposition theorem. Then apply the same theorem to $\phi(1_i, \boldsymbol{x})$ and $\phi(0_i, \boldsymbol{x})$.

Series and parallel components

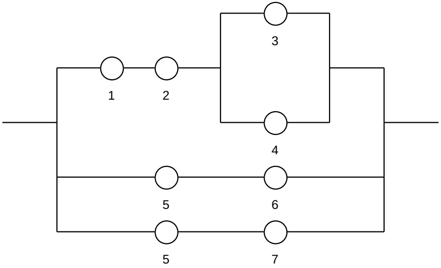
Definition

Let (C, ϕ) be a binary monotone system, and let $i, j \in C$.

We say that i and j are *in series* if ϕ depends on the component state variables, x_i and x_j, only through the product $x_i \cdot x_j$.

We say that i and j are *in parallel* if ϕ depends on the component state variables, x_i and x_j, only through the coproduct $x_i \amalg x_j$.

Series and parallel components (cont.)



In this system components 1 and 2 are in series, while components 3 and 4 are in parallel. Note, however, that components 5 and 6 are *not* in series since component 5 is also connected via component 7. Moreover, components 6 and 7 are in parallel.

Series and parallel components (cont.)

Theorem

Let (C, ϕ) be a binary monotone system, and let $i, j \in C$. Moreover, assume that the component state variables are independent.

If i and j are in series, then the reliability function, h, depends on p_i and p_j only through $p_i \cdot p_j$.

If i and j are in parallel, then the reliability function, h, depends on p_i and p_j only through $p_i \amalg p_j$.

Series and parallel components (cont.)

PROOF: If i and j are in series, we have:

$$\phi(1_i, 0_j, \boldsymbol{x}) = \phi(0_i, 1_j, \boldsymbol{x}) = \phi(0_i, 0_j, \boldsymbol{x}).$$

Thus, by pivotal decomposition we have:

$$\begin{aligned}\phi(\boldsymbol{x}) &= x_i x_j \phi(1_i, 1_j, \boldsymbol{x}) + x_i(1 - x_j)\phi(1_i, 0_j, \boldsymbol{x}) \\ &+ (1 - x_i)x_j\phi(0_i, 1_j, \boldsymbol{x}) + (1 - x_i)(1 - x_j)\phi(0_i, 0_j, \boldsymbol{x}) \\ &= (x_i x_j) \cdot \phi(1_i, 1_j, \boldsymbol{x}) + (1 - (x_i x_j)) \cdot \phi(0_i, 0_j, \boldsymbol{x}).\end{aligned}$$

Hence, by replacing the vector $\boldsymbol{x}$ by $\boldsymbol{X}$ and taking expectations we get:

$$h(\boldsymbol{p}) = (p_i p_j) \cdot h(1_i, 1_j, \boldsymbol{p}) + (1 - (p_i p_j)) \cdot h(0_i, 0_j, \boldsymbol{p}).$$

That is, h, depends on p_i and p_j only through $p_i \cdot p_j$.

Series and parallel components (cont.)

If i and j are in parallel, we have:

$$\phi(1_i, 1_j, \boldsymbol{x}) = \phi(1_i, 0_j, \boldsymbol{x}) = \phi(0_i, 1_j, \boldsymbol{x}).$$

Thus, by pivotal decomposition we have:

$$\begin{aligned} \phi(\boldsymbol{x}) &= x_i x_j \phi(1_i, 1_j, \boldsymbol{x}) + x_i(1 - x_j)\phi(1_i, 0_j, \boldsymbol{x}) \\ &\quad + (1 - x_i)x_j\phi(0_i, 1_j, \boldsymbol{x}) + (1 - x_i)(1 - x_j)\phi(0_i, 0_j, \boldsymbol{x}) \\ &= (x_i \amalg x_j) \cdot \phi(1_i, 1_j, \boldsymbol{x}) + (1 - (x_i \amalg x_j)) \cdot \phi(0_i, 0_j, \boldsymbol{x}). \end{aligned}$$

Hence, by replacing the vector $\boldsymbol{x}$ by $\boldsymbol{X}$ and taking expectations we get:

$$h(\boldsymbol{p}) = (p_i \amalg p_j) \cdot h(1_i, 1_j, \boldsymbol{p}) + (1 - (p_i \amalg p_j)) \cdot h(0_i, 0_j, \boldsymbol{p}).$$

That is, h, depends on p_i and p_j only through $p_i \amalg p_j$.

s-p-reductions

Consider a binary monotone system, (C, ϕ) where the component state variables are independent, and let $i, j \in C$.

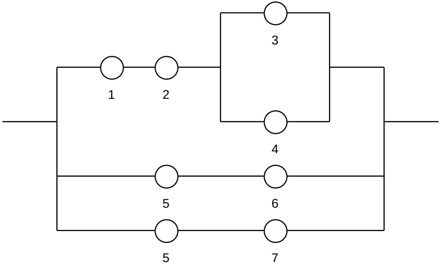
SERIES REDUCTION: If the components i and j are in series, then we may replace i and j by a single component i' with reliability $p_{i'} = p_i p_j$ without altering the system reliability.

PARALLEL REDUCTION: If the components i and j are in parallel, then we may replace i and j by a single component i' with reliability $p_{i'} = p_i \amalg p_j$ without altering the system reliability.

Series and parallel reductions are referred to as *s-p-reductions*. Each s-p-reduction reduces the number of components in the system by one.

A system that can be reduced to a single component by applying a sequence of s-p-reductions is called an *s-p-system*.

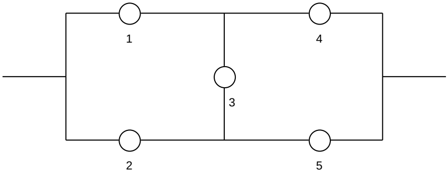
s-p-reductions (cont.)



This 7-component system is an s-p-system. Its reliability function can be derived using s-p-reductions *only* and is given by:

$$h(\boldsymbol{p}) = [p_1 p_2 (p_3 \amalg p_4)] \amalg [p_5 (p_6 \amalg p_7)]$$

Pivotal decompositions and s-p-reductions

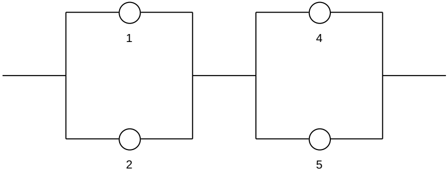


Let (C, ϕ) be the *bridge structure* shown above. In order to derive the structure function of this system, we note that:

$$\phi(1_3, \boldsymbol{X}) = \text{The system state given that component 3 is functioning}$$

$$\phi(0_3, \boldsymbol{X}) = \text{The system state given that component 3 is failed}$$

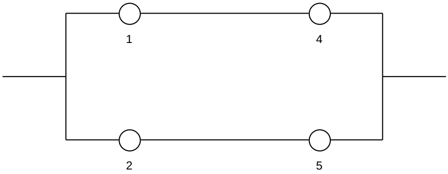
Pivotal decompositions and s-p-reductions (cont.)



Given that component 3 is functioning, the system becomes a series connection of two parallel systems. Hence, by using s-p-reductions, we get that:

$$\phi(1_3, \boldsymbol{X}) = (X_1 \amalg X_2) \cdot (X_4 \amalg X_5).$$

Pivotal decompositions and s-p-reductions (cont.)



Given that component 3 is failed, the system becomes a parallel connection of two series systems. Hence, by using s-p-reductions, we get that:

$$\phi(0_3, \boldsymbol{X}) = (X_1 \cdot X_4) \amalg (X_2 \cdot X_5).$$

Pivotal decompositions and s-p-reductions (cont.)

By the pivotal decomposition theorem it follows that ϕ can be written as:

$$\phi(\boldsymbol{X}) = X_3 \cdot \phi(1_3, \boldsymbol{X}) + (1 - X_3) \cdot \phi(0_3, \boldsymbol{X}).$$

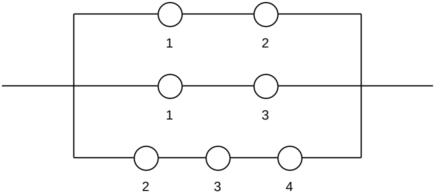
Combining all this we get that ϕ is given by:

$$\phi(\boldsymbol{X}) = X_3 \cdot (X_1 \amalg X_2)(X_4 \amalg X_5) + (1 - X_3) \cdot (X_1 \cdot X_4 \amalg X_2 \cdot X_5).$$

Moreover, assuming that the component state variables are independent, the reliability of the systems is:

$$h(\boldsymbol{p}) = p_3 \cdot (p_1 \amalg p_2)(p_4 \amalg p_5) + (1 - p_3) \cdot (p_1 \cdot p_4 \amalg p_2 \cdot p_5).$$

Pivotal decompositions and s-p-reductions

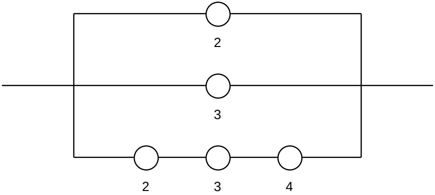


Let (C, ϕ) be the system shown above. In order to derive the structure function of this system, we note that:

$$\phi(1_1, \boldsymbol{X}) = \text{The system state given that component 1 is functioning}$$

$$\phi(0_1, \boldsymbol{X}) = \text{The system state given that component 1 is failed}$$

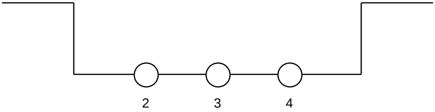
Pivotal decompositions and s-p-reductions (cont.)



Given that component 1 is functioning, the system becomes a parallel system of components 2 and 3 (since the lower path $\{2, 3, 4\}$ can be ignored in this case). Hence, by using s-p-reductions, we get that:

$$\phi(1_1, \boldsymbol{X}) = X_2 \amalg X_3.$$

Pivotal decompositions and s-p-reductions (cont.)



Given that component 1 is failed, the system becomes a series system of components 2, 3 and 4. Hence, by using s-p-reductions, we get that:

$$\phi(0_1, \boldsymbol{X}) = X_2 \cdot X_3 \cdot X_4.$$

Pivotal decompositions and s-p-reductions (cont.)

By the pivotal decomposition theorem it follows that ϕ can be written as:

$$\phi(\boldsymbol{X}) = X_1 \cdot \phi(1_1, \boldsymbol{X}) + (1 - X_1) \cdot \phi(0_1, \boldsymbol{X}).$$

Combining all this we get that ϕ is given by:

$$\phi(\boldsymbol{X}) = X_1 \cdot (X_2 \amalg X_3) + (1 - X_1) \cdot (X_2 \cdot X_3 \cdot X_4).$$

Moreover, assuming that the component state variables are independent, the reliability of the systems is:

$$h(\boldsymbol{p}) = p_1 \cdot (p_2 \amalg p_3) + (1 - p_1) \cdot (p_2 \cdot p_3 \cdot p_4).$$

Strict monotonicity

Theorem

Let $h(\boldsymbol{p})$ be the reliability function of a binary monotone system (C, ϕ) of order n, and assume that $0 < p_j < 1$ for all $j \in C$. If component i is relevant, then $h(\boldsymbol{p})$ is strictly increasing in p_i.

PROOF: Using pivotal decomposition wrt. component *i* it follows that:

$$\frac{\partial h(\boldsymbol{p})}{\partial p_i} = \frac{\partial}{\partial p_i}[p_i h(1_i, \boldsymbol{p}) + (1 - p_i) h(0_i, \boldsymbol{p})]$$

$$= h(1_i, \boldsymbol{p}) - h(0_i, \boldsymbol{p})$$

$$= E[\phi(1_i, \boldsymbol{X})] - E[\phi(0_i, \boldsymbol{X})] = E[\phi(1_i, \boldsymbol{X}) - \phi(0_i, \boldsymbol{X})]$$

$$= \sum_{(\cdot_i, \boldsymbol{x}) \in \{0,1\}^{n-1}} \left[\phi(1_i, \boldsymbol{x}) - \phi(0_i, \boldsymbol{x})\right] P\big((\cdot_i, \boldsymbol{X}) = (\cdot_i, \boldsymbol{x})\big)$$

Strict monotonicity (cont.)

Since ϕ is non-decreasing in each argument it follows that:

$$\big[\phi(1_i, \boldsymbol{x}) - \phi(0_i, \boldsymbol{x})\big] \geq 0, \text{ for all } (\cdot_i, \boldsymbol{x}) \in \{0, 1\}^{n-1}.$$

If i is relevant, there exists at least one $(\cdot_i, \boldsymbol{y}) \in \{0, 1\}^{n-1}$ such that:

$$\big[\phi(1_i, \boldsymbol{y}) - \phi(0_i, \boldsymbol{y})\big] > 0.$$

Since $0 < p_j < 1$ for all $j \in C$, we have:

$$P\big((\cdot_i, \boldsymbol{X}) = (\cdot_i, \boldsymbol{x})\big) > 0, \text{ for all } (\cdot_i, \boldsymbol{x}) \in \{0, 1\}^{n-1}.$$

From this it follows that:

$$\frac{\partial h(\boldsymbol{p})}{\partial p_i} > 0.$$

That is, $h(\boldsymbol{p})$ is strictly increasing in p_i.